Probabilistic Programming Languages: Independent Choices and Deterministic Systems

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. . . the way causal models were first introduced into genetics, econometrics, and the social sciences, as well as . . . the way causal models are used routinely in physics end engineering . . . causal relationships are expressed in the form of deterministic functional equations, and probabilities are introduced through the assumption that certain variables in the equations are unobserved. This reflects Laplace's (1814) conception of natural phenomena, according to which nature's laws are deterministic and randomness surfaces owing merely to our ignorance of the underlying boundary conditions. . . .

. . . we shall express preference towards Laplace's quasi-deterministic conception of causality. . .

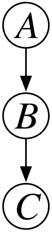
—Judea Pearl [2000] page 26

Outline

Probabilistic Programming Languages

- Probabilistic inputs (used in Simula in 1966)
- Conditioning on observations, and querying for distributions
- Inference: more efficient than rejection sampling
- Learning probabilities from data

Representing Bayesian networks



$$
\begin{aligned}
P(a) &= 0.1 \\
P(b \mid a) &= 0.8 \\
P(b \mid \neg a) &= 0.3 \\
P(c \mid b) &= 0.4 \\
P(c \mid \neg b) &= 0.75
\end{aligned}
$$

$P(a) = 0.1,$
$P(bifa) = 0.8,\ P(bifna) = 0.3,$
$P(cifb) = 0.4,\ P(cifnb) = 0.75.$
$b \iff (a \wedge bifa) \vee (\neg a \wedge bifna)$
$c \iff (b \wedge cifb) \vee (\neg b \wedge cifnbc)$

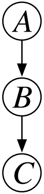
```
begin
    Boolean a,b,c;
    a := draw(0.1);
    if a then
        b := draw(0.8);
    else
        b := draw(0.3);
    if b then
        c := draw(0.4);
    else
        c := draw(0.75);
end
```

Semantics of Probabilistic Programming Languages

Choices among alternatives are independent.
Program specifies the consequences of choices.

- Rejection sampling: probability of a proposition is the proportion of samples that generate that proposition
- Independent choice: possible world for each assignment of a value for each alternative; program specifies what is true in each world
- Program trace semantics: possible world for each choice encountered in execution path
- Abductive semantics: measure over independent choice worlds; only make distinctions needed to answer a query — provides a measure space over the independent choices

Independent Choice Semantics



$$P(a) = 0.1, P(bifa) = 0.8,\ P(bifna) = 0.3,$$
$$P(cifb) = 0.4,\ P(cifnb) = 0.75.$$
$$b \iff (a \land bifa) \lor (\neg a \land bifna)$$
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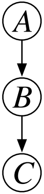
World	*A*	*Bifa*	*Bifna*	*Cifb*	*Cifnb*	Probability
w_0	false	false	false	false	false	$0.9 \cdot 0.2 \cdot 0.7 \cdot 0.6 \cdot 0.25$
w_1	false	false	false	false	true	$0.9 \cdot 0.2 \cdot 0.7 \cdot 0.6 \cdot 0.75$
...						
w_{30}	true	true	true	true	false	$0.1 \cdot 0.8 \cdot 0.3 \cdot 0.4 \cdot 0.75$
w_{31}	true	true	true	true	true	$0.1 \cdot 0.8 \cdot 0.3 \cdot 0.4 \cdot 0.75$

Program Trace Semantics



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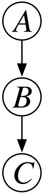
World	A	$Bifa$	$Bifna$	$Cifb$	$Cifnb$	Probability
w_0	false	$\perp$	false	$\perp$	false	$0.9 \times 0.7 \times 0.25$
w_1	false	$\perp$	false	$\perp$	true	$0.9 \times 0.7 \times 0.75$
...						
w_6	true	true	$\perp$	false	$\perp$	$0.1 \times 0.8 \times 0.6$
w_7	true	true	$\perp$	true	$\perp$	$0.1 \times 0.8 \times 0.4$

Program Trace Semantics



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Abductive semantics — these are sets of independent choice worlds

Semantics Example

```
begin
   Boolean x;
   x := draw(0.2);
   if x then
      begin
        Boolean y;
        y := draw(0.5);
        ...
      end
   else
      begin
        Boolean z;
        z := draw(0.7);
        ...
      end
```

- y only defined when x is true.
- z only defined when x is false.

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- y only defined when x is true.
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- Program trace semantics: y and z are not defined in the same possible worlds.
- Independent choice semantics: the choices are all independent of each other.
- Abductive semantics: worlds which only differ by untaken choices are grouped together.
- What program transformations are legal?

Semantics Example

```
begin
   Integer i;
   i := 1;
   while (True)
      begin
        Boolean x;
        x := draw(0.01);
        if x then
           return i;
        else
           i := i+1;
      end
end
```

- What is the expected value of i?

Semantics Example

```
begin
   Integer i;
   i := 1;
   while (True)
      begin
        Boolean x;
        x := draw(0.01);
        if x then
           return i;
        else
           i := i+1;
      end
end
```

- What is the expected value of i?
- How many independent choice worlds are there?

Semantics Example

```
begin
   Integer i;
   i := 1;
   while (True)
      begin
        Boolean x;
        x := draw(0.01);
        if x then
           return i;
        else
           i := i+1;
      end
end
```

- What is the expected value of i?
- How many independent choice worlds are there?
- What is the probability of the most likely one?

Semantics Example

```
begin
   Integer i;
   i := 1;
   while (True)
      begin
        Boolean x;
        x := draw(0.01);
        if x then
           return i;
        else
           i := i+1;
      end
end
```

- What is the expected value of i?
- How many independent choice worlds are there?
- What is the probability of the most likely one?
- program choice semantics: choices not made are undefined
- abductive semantics: worlds that only differ in choices not made are grouped together

Semantics and Inference

```
begin
   Boolean x;
   x := draw(0.2);
   if x then
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        Boolean y;
        y := draw(0.5);
        ...
      end
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```

- y is only defined when x is true.
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- In variable elimination, what happens when x is summed out?

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      end
```

- y is only defined when x is true.
- z is only defined when x is false.
- In variable elimination, what happens when x is summed out?
- In MCMC, what happens when x has its value changed?

Semantics

```
Boolean x;
x := draw(0.2);
if x then
   return 1;
else
   begin
     x := draw(0.5);
     if x then
        return 2;
     else
        return 3;
```

- What is the probability 1 is returned?

Semantics

```
Boolean x;
x := draw(0.2);
if x then
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   begin
     x := draw(0.5);
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     else
        while (True)
           begin

           end
        return 3;
```

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     if x then
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     else
        while (True)
           begin
              x := draw(0.3)
           end
        return 3;
```

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     else
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           begin
              x := draw(0.3)
           end
        return 1;
```

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Boolean x;
x := draw(0.2);
if x then
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   begin
     x := draw(0.5);
     if x then
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     else
        if p_equals_np() then
           return 3;
        else
           return 4;
   end
```

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           return 2;
   end
```

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Outline

Observing

- What happens when the vocabulary used in models does not match the vocabulary of observations?
- How can we specify the observations so they interact with programs?
- What happens when observational data and models are build by diverse sets of people?

Probability of an observation

- Given a model of rooms of houses and their colours:
- A person observes a house and reports:
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  - They just finished painting the kitchen.

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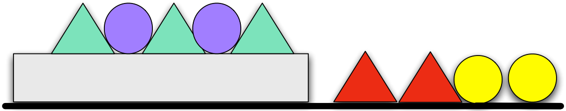
  - They picked a room at random and reported its colour.
  - They told us the colour of all of the rooms.
  - They searched for a room that is green and reported that they found the kitchen was green.
  - This was the most interesting/unusual aspect of the house.
  - They just finished painting the kitchen.
- The probability depends on the protocol for observations.

Observation Protocols

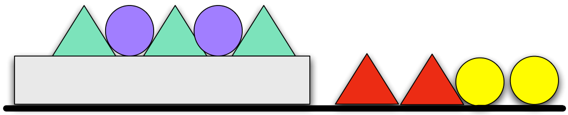


Observe a triangle and a circle touching. What is the probability the triangle is green?

$$P(green(x) \mid triangle(x) \land \exists y \; circle(y) \land touching(x, y))$$

The answer depends on how the x and y were chosen!

Protocol for Observing



$$P(green(x) \mid triangle(x) \wedge \exists y\ circle(y) \wedge touching(x,y))$$

$select(x)$	$select(y)$	$select(x,y)$
$select(y)$	$select(x)$	
$3/4$	$2/3$	$4/5$

Apartment/House Domain

Given:

- a database of descriptions of apartments and houses available to rent.
- a set of programs that predict what house a person would be happy with. Each specifies $P(person\_likes \mid description)$.

Want:

- for each house determine which person would most likely want it
- for each person determine which house they would be most likely to like.

Role assignments

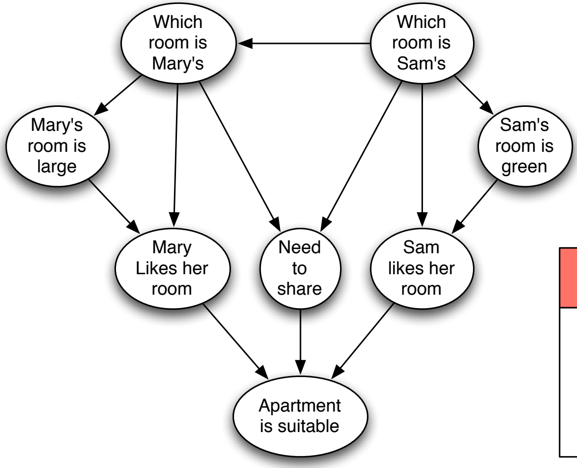
Hypothesis about what apartment Mary would like.

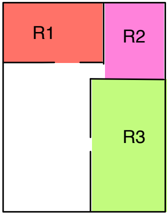
Whether Mary likes an apartment depends on:

- Whether there is a bedroom for daughter Sam
- Whether Sam's room is green
- Whether there is a bedroom for Mary
- Whether Mary's room is large
- Whether they share

... but apartments don't come labelled with the roles.

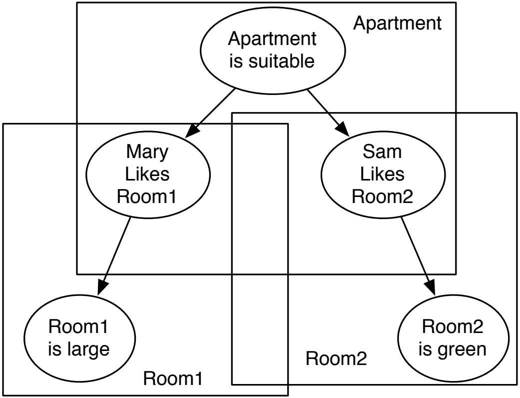
Bayesian Belief Network Representation

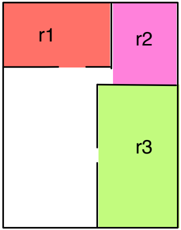




How can we condition on the observation of the apartment?

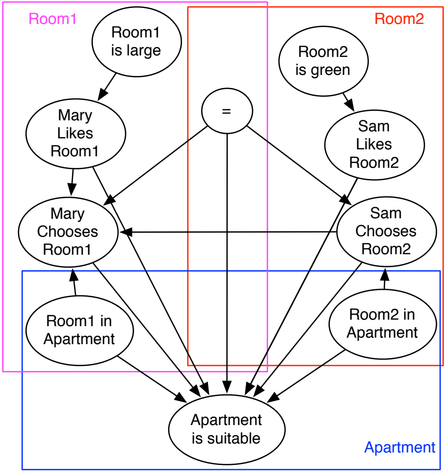
Naive Bayes representation

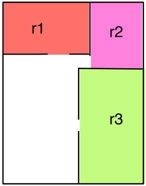




How do we specify that Mary chooses a room?
What about the case where they (have to) share?

Causal representation





How do we specify that Sam and Mary choose one room each, but they can like many rooms?

Conclusion

- Probabilistic programming language: independent probabilistic choices + deterministic programming language (logic programming, ML, Scheme, Java, C,. . .)
- Need observation languages to complement probabilistic programming languages.
- Many challenges:
    - inference
    - learning
    - conditioning on all relevant data (available anywhere in the world)
    - heterogeneous data sets and semantic interoperability
    - heterogeneous probabilistic models (multiple levels of abstraction and detail)
    - probability of identity and existence