A Toolbox of Level Set Methods

Ian Mitchell
Department of Computer Science
University of British Columbia

http://www.cs.ubc.ca/~mitchell
mitchell@cs.ubc.ca

research supported by
the Natural Science and Engineering Research Council of Canada

Level Set Methods

- Numerical algorithms for dynamic implicit surfaces and Hamilton-Jacobi partial differential equations
- Applications in
  - Graphics, Computational Geometry and Mesh Generation
  - Differential Games
  - Financial Mathematics and Stochastic Differential Equations
  - Fluid and Combustion Simulation
  - Image Processing and Computer Vision
  - Robotics, Control and Dynamic Programming
  - Verification and Reachable Sets

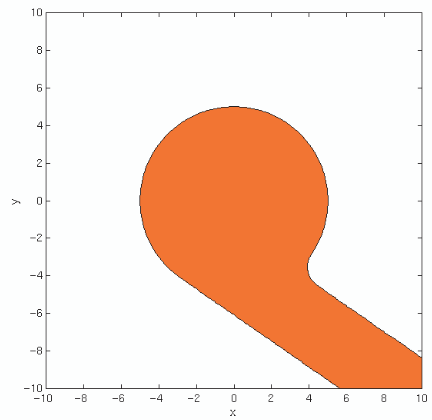
Implicit Surface Functions

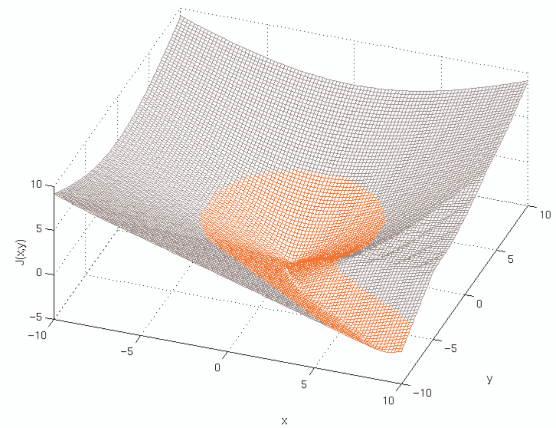
- Surface $S(t)$ and/or set $G(t)$ are defined implicitly by an isosurface of a scalar function $\phi(x,t)$, with several benefits
  - State space dimension does not matter conceptually
  - Surfaces automatically merge and/or separate
  - Geometric quantities are easy to calculate

$$\phi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$$

$$\mathcal{G}(t) = \{x \in \mathbb{R}^n \mid \phi(x,t) \leq 0\}$$

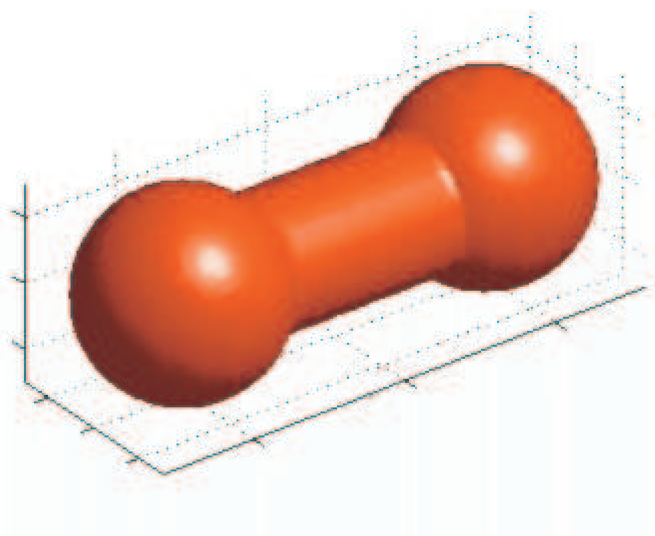
$$\mathcal{S}(t) = \partial \mathcal{G}(t) = \{x \in \mathbb{R}^n \mid \phi(x,t) = 0\}$$



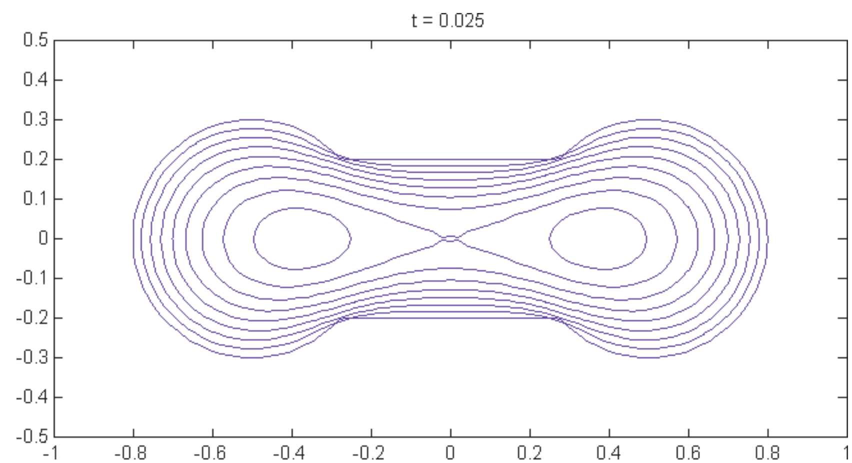


Implicit Surface Benefits

- Easy to represent a variety of shapes
- Unified framework for many types of motion
- Surface parameters easily approximated
- Topological changes are automatic
- Conceptual complexity independent of dimension
- Easy to visualize
- Easy to implement (?)



shrinking dumbbell



contour slice through midplane

Hamilton-Jacobi Equations

$$D_t\varphi(x,t) + G(x,t,\varphi,\nabla\varphi,D_x^2\varphi) = 0$$

$$\varphi(x,0) = g(x) \text{ bounded and continuous}$$

$$G(x,t,r,p,\mathbf{X}) \leq G(x,t,s,p,\mathbf{Y}), \text{ if } r \leq s \text{ and } \mathbf{Y} \leq \mathbf{X}$$

- Time-dependent partial differential equation (PDE)
  - With second derivative terms: degenerate hyperbolic PDE
- In general, classical solution will not exist
  - Viscosity solution φ will be continuous but not differentiable
- For example, classical Hamilton-Jacobi-Bellman
  - Finite horizon optimal cost leads to terminal value PDE

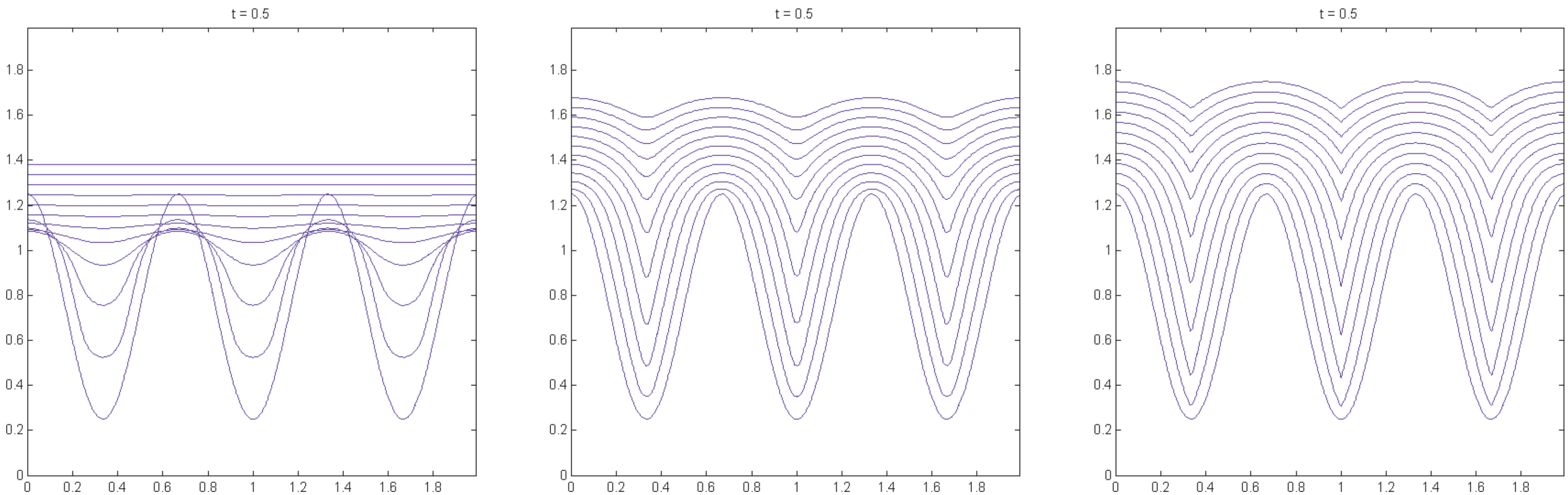
$$\varphi(x(t),t) = \min_{u(\cdot)} \left[g(x(T)) + \int_t^T \ell(x(s),u(s))ds \right]$$

$$D_t\varphi(x,t) + \min_u \left[\nabla\varphi(x,t)\cdot f(x,u) + \ell(x,u)\right] = 0$$

Viscosity Solution

- Well defined weak solution of HJ PDE
  - Limit of vanishing viscosity solution, where it exists
  - Kinks form where characteristics cross
- Example

$$D_t\phi(x,t) + (1 - b\kappa(x,t))\|\nabla\phi(x,t)\| = 0$$



$b = 0.25$ $b = 0.025$ $b = 0$

The Toolbox: What Is It?

- A collection of Matlab routines for level set methods
  - Fixed Cartesian grids
  - Arbitrary dimension (computationally limited)
  - Vectorized code achieves reasonable speed
  - Direct access to Matlab debugging and visualization
  - Source code is provided for all toolbox routines
- Underlying algorithms
  - Solve various forms of Hamilton-Jacobi PDE
  - First and second spatial derivatives
  - First temporal derivatives
  - High order accurate approximation schemes
  - Explicit temporal integration

The Toolbox: What Can It Do?

$$
\begin{aligned}
0 = & D_t \phi(x,t) && \text{temporal derivative} \\
& + v(x,t) \cdot \nabla \phi(x,t) && \text{convection} \\
& + a(x,t) \|\nabla \phi(x,t)\| && \text{normal motion} \\
& + \text{sign}(\phi(x,0))(\|\nabla \phi(x,t)\| - 1) && \text{reinitialization} \\
& + H(x,t,\phi,\nabla \phi) && \text{general HJ} \\
& - b(x,t)\kappa(x,t)\|\nabla \phi(x,t)\| && \text{mean curvature} \\
& - \text{trace}[\mathbf{L}(x,t) D_x^2 \phi(x,t) \mathbf{R}(x,t)] && \text{stochastic DEs} \\
& + \lambda(x,t)\phi(x,t) && \text{discounting} \\
& + F(x,t,\phi), && \text{forcing}
\end{aligned}
$$

$$
\begin{aligned}
D_t\phi(x,t) &\geq 0, & D_t\phi(x,t) &\leq 0, && \text{growth constraints} \\
\phi(x,t) &\leq \psi(x,t), & \phi(x,t) &\geq \psi(x,t), && \text{masking constraints}
\end{aligned}
$$

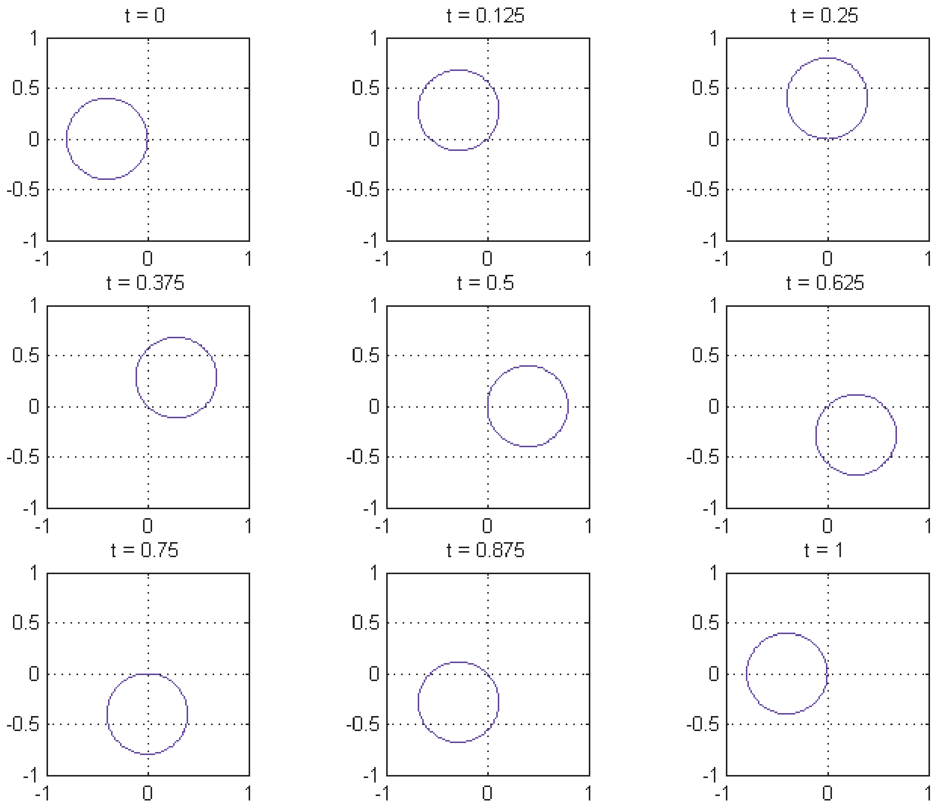
$\phi : \mathbb{R}^n \to \mathbb{R}^m$ vector level sets

Convective Flow

- Motion by externally generated velocity field

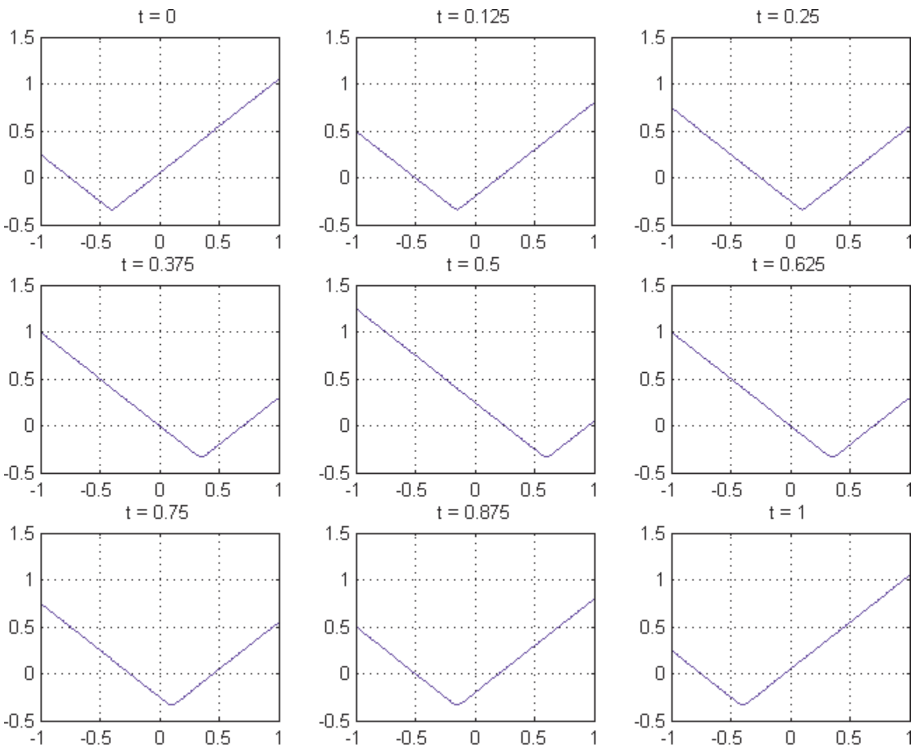
$$D_t\phi(x,t) + v(x,t) \cdot \nabla\phi(x,t) = 0$$

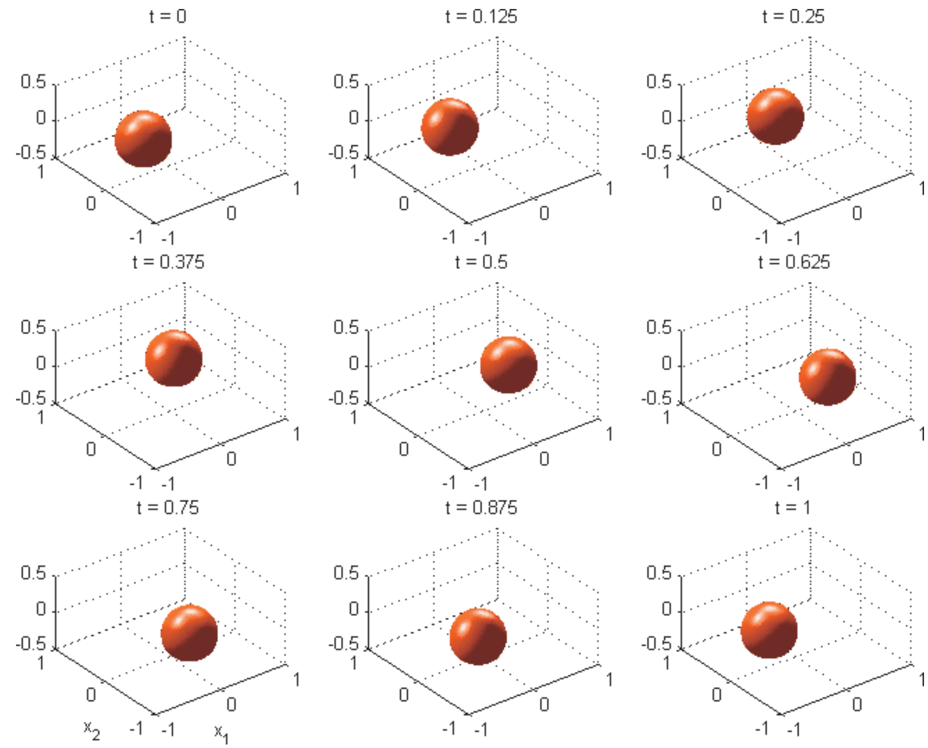
- Example: rigid body rotation about the origin



Dimensionally Flexible

- Core code is dimensionally independent
  - Cost in memory and computation is exponential
  - Visualization in dimensions four and above is challenging
  - Dimensions one to three are quite feasible

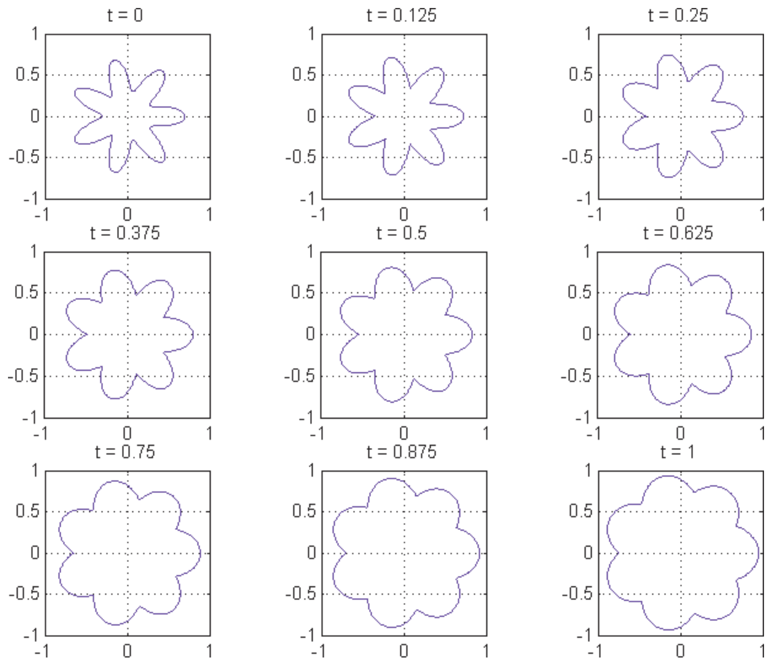




Motion in the Normal Direction

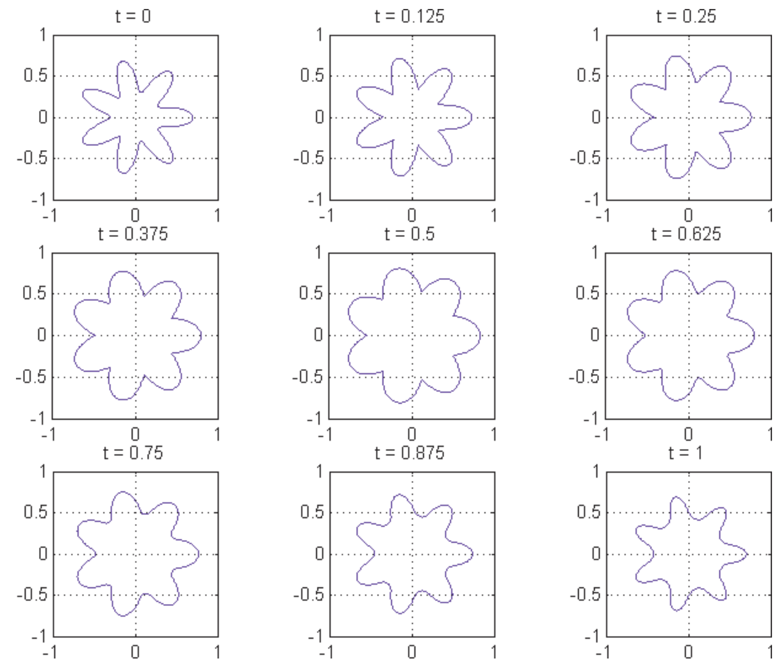
- Motion by externally generated speed function

$$D_t\phi(x,t) + a(x,t)\|\nabla\phi(x,t)\| = 0$$



constant outward speed

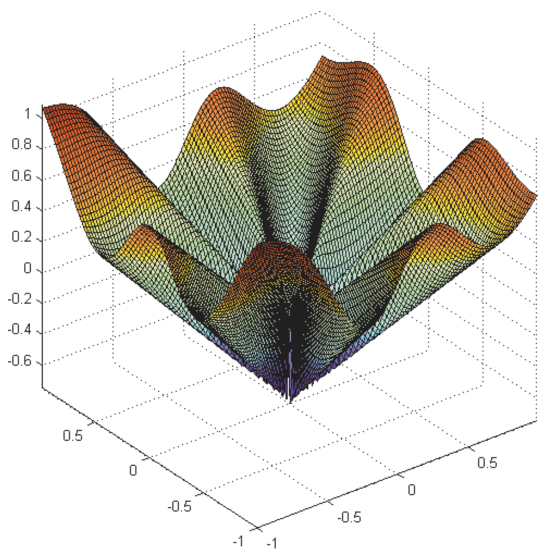
constant speed switches direction outward at first, inward thereafter



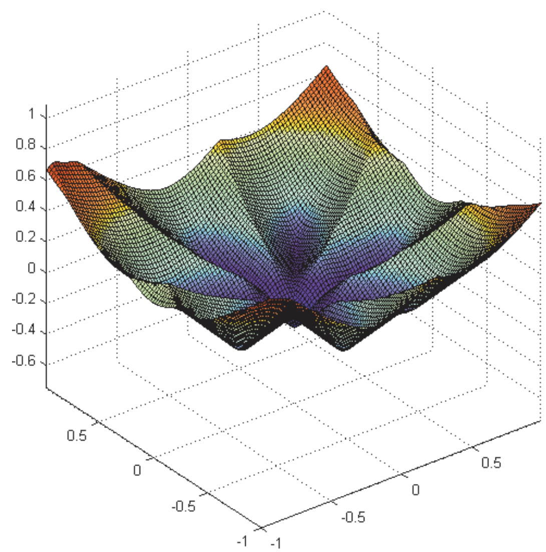
Reinitialization Equation

- Returning the gradient to unit magnitude

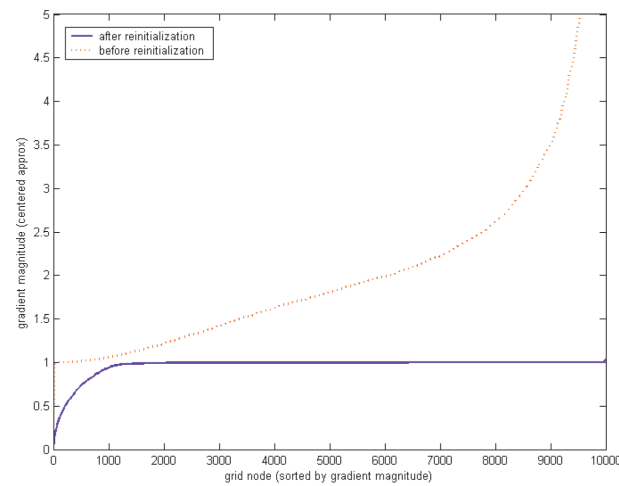
$$D_t\phi(x,t) + \mathrm{sign}(\phi(x,0))(\|\nabla\phi(x,t)\| - 1) = 0$$



initial implicit surface function



reinitialized to signed distance



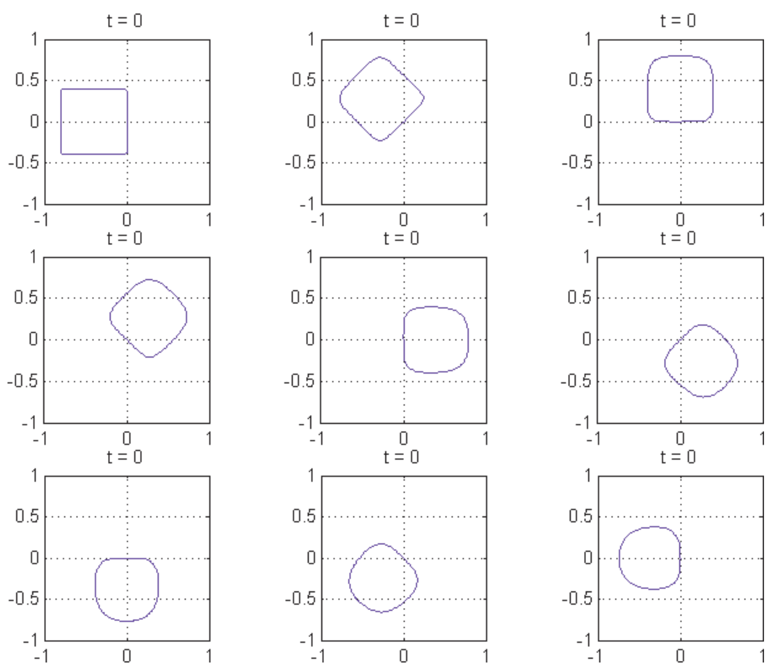
compare gradient magnitudes

General Hamilton-Jacobi

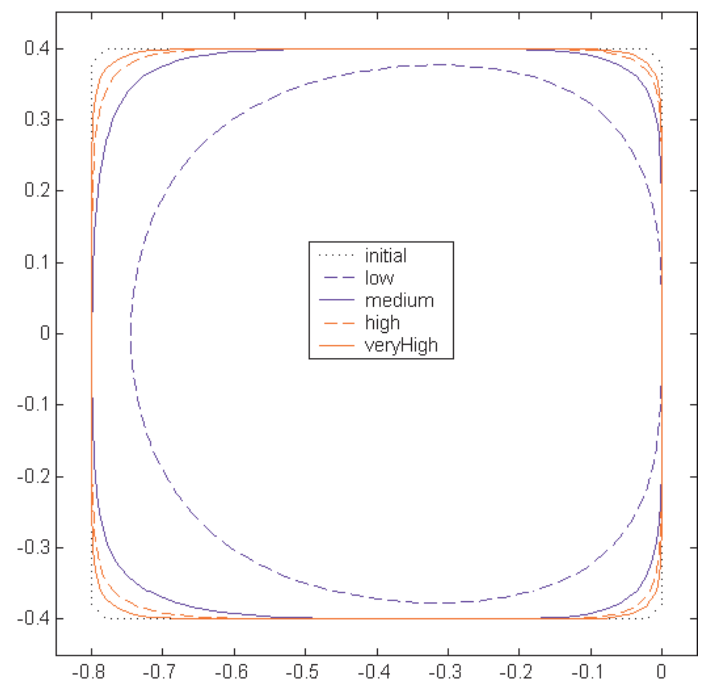
- Motion may depend nonlinearly on gradient

$$D_t \phi(x,t) + H(x, t, \nabla \phi(x,t)) = 0$$

- Example: rigid body rotation about the origin



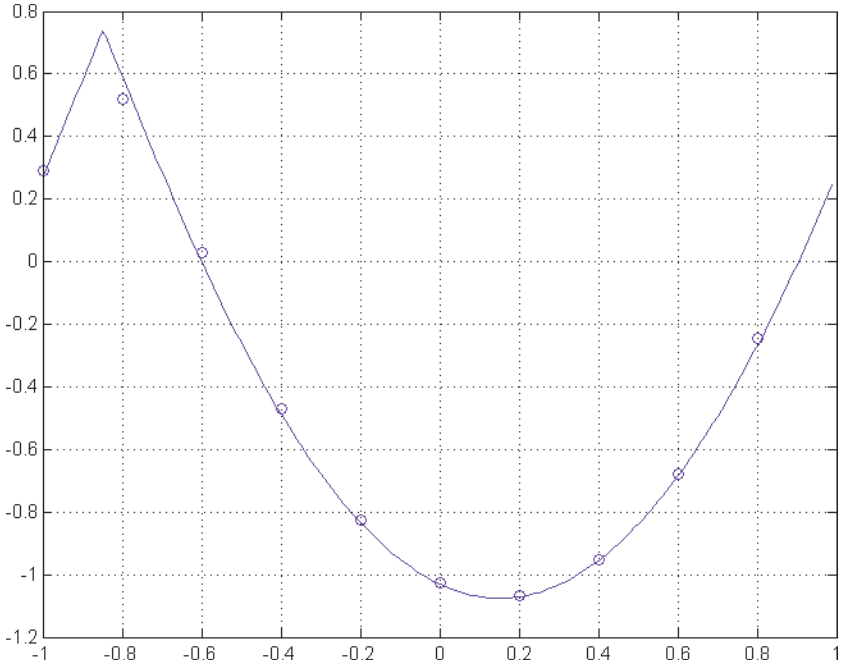
rotate a square once around

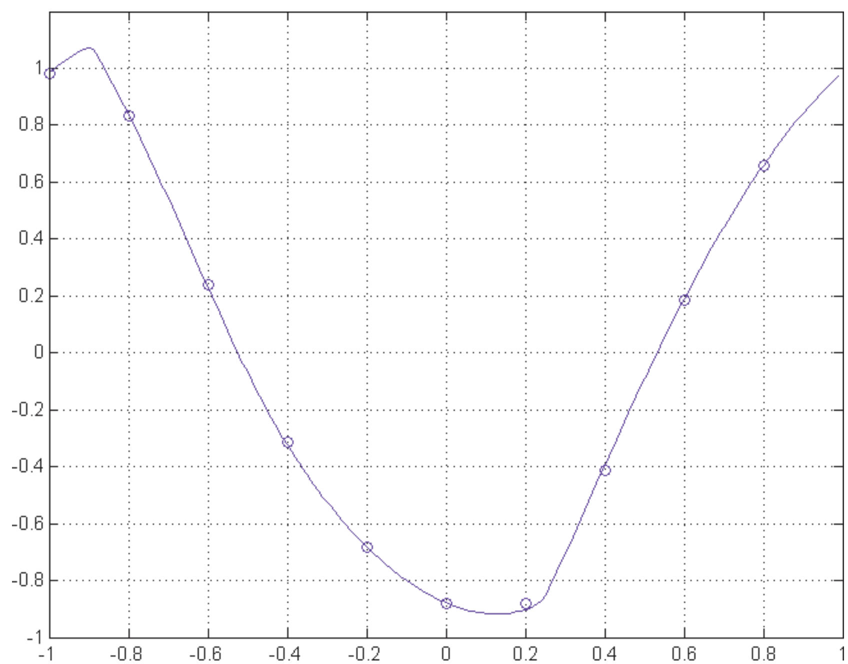


compare errors of various schemes

General Hamilton-Jacobi

$$D_t\phi(x,t) + H(x,t,\nabla\phi(x,t)) = 0$$





$$H(x,t,p) = \frac{1}{2}\left(\alpha + \sum_i p_i\right)^2$$

Burgers' equation

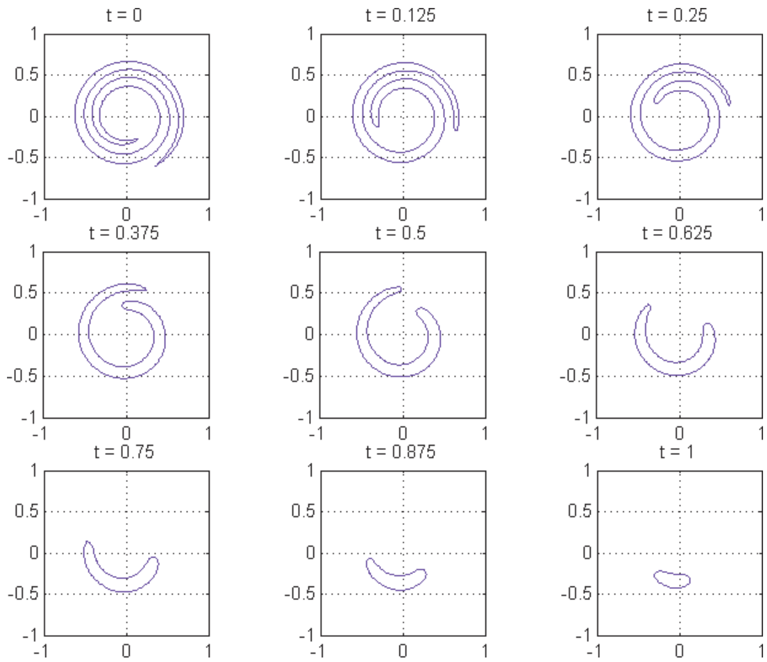
$$H(x,t,p) = -\cos\left(\alpha + \sum_i p_i\right)$$

Nonconvex Hamiltonian

Motion by Mean Curvature

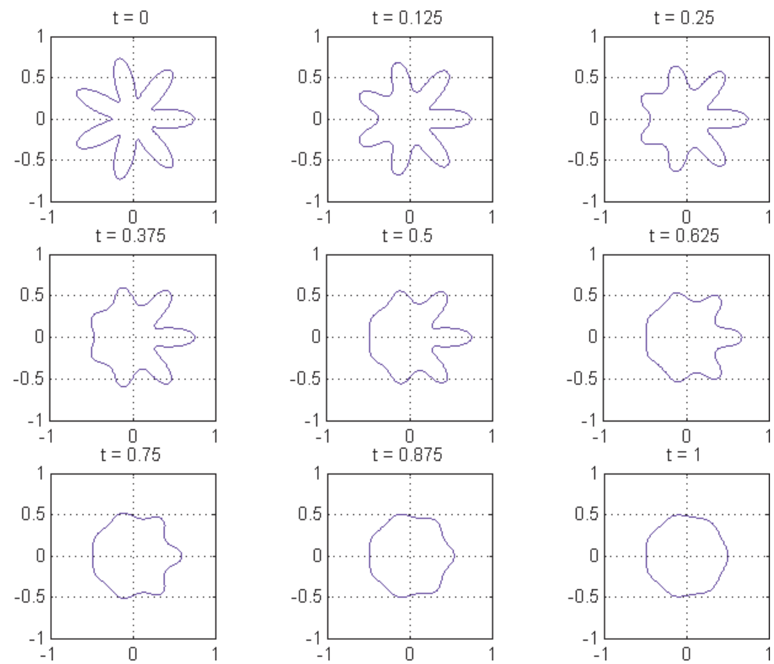
- Interface speed depends on its curvature κ

$$D_t\phi(x,t) - b(x,t)\kappa(x,t)\|\phi(x,t)\| = 0$$



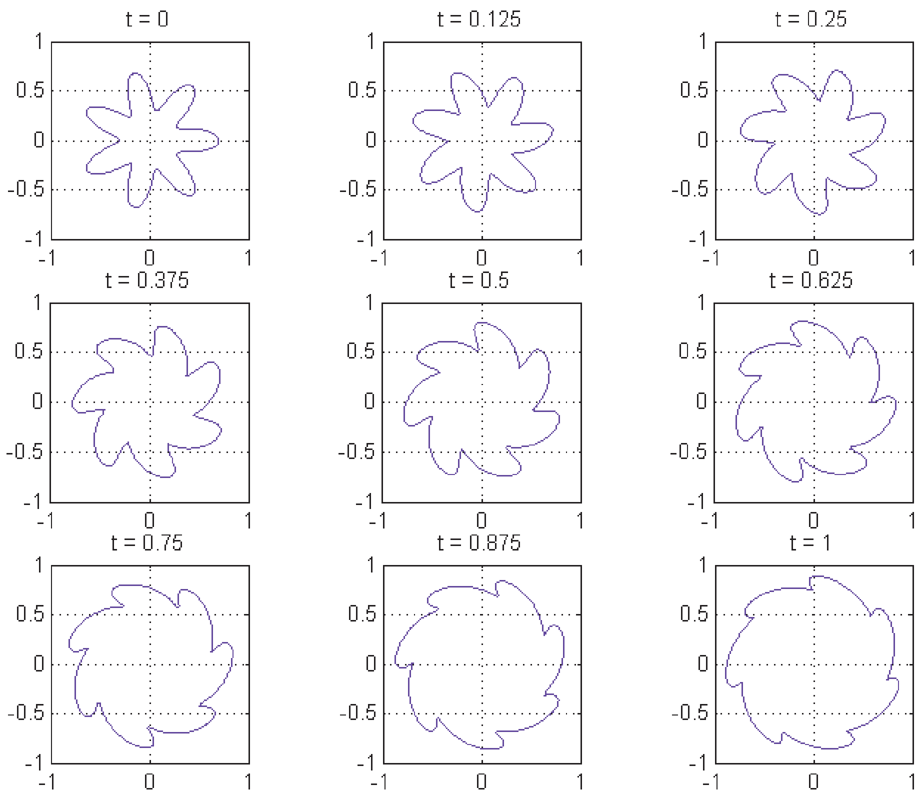
shrinking spiral

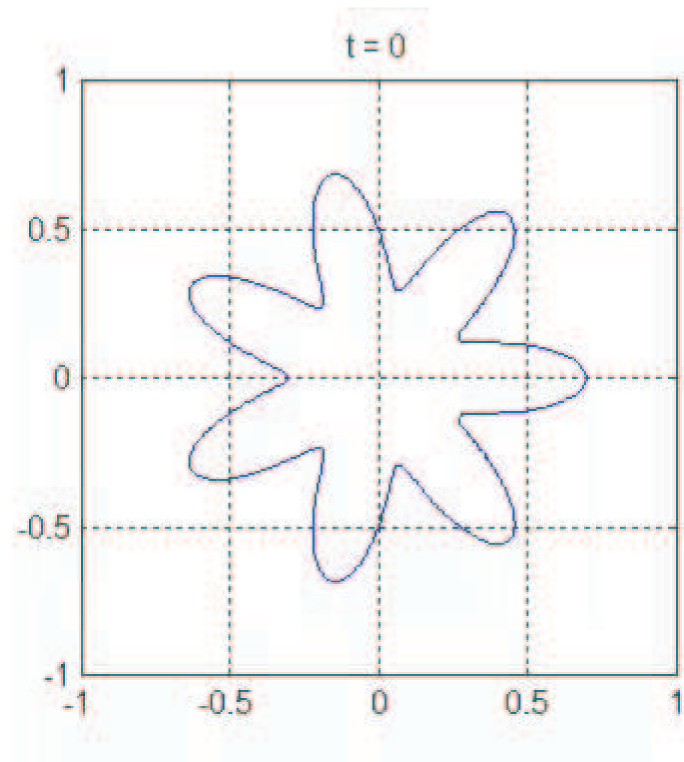
shrinking star
speed depends on time



Combining Terms

- Terms can be combined to generate complex but accurate motion
  - Example: rotation plus outward motion in normal direction

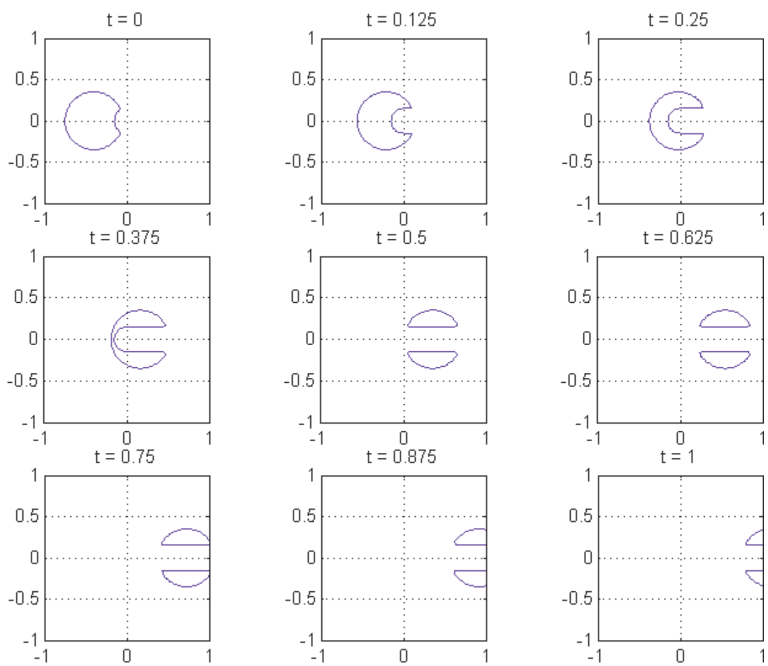




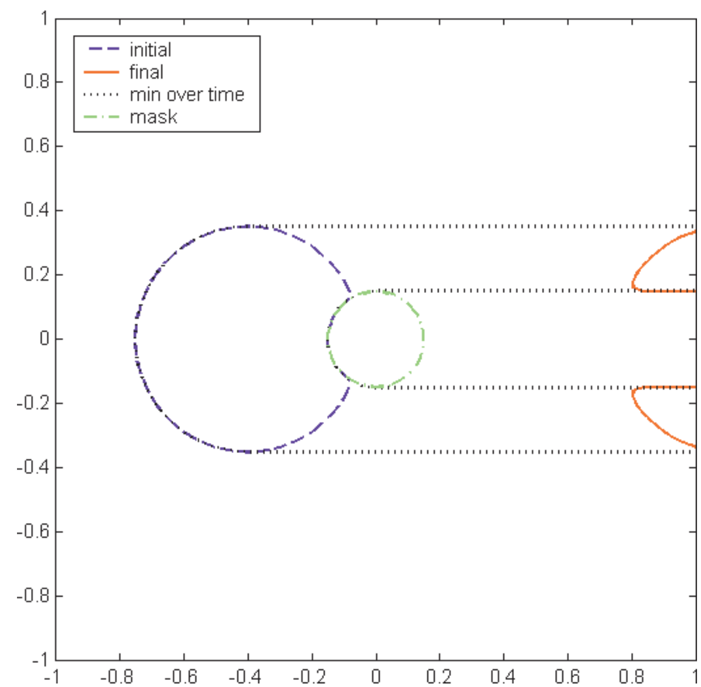
Constraints on Function Value

- Level set function constrained by user supplied implicit surface function
  - Example: masking a region of the state space

$$\phi(x,t) \leq \psi(x) \qquad \phi(x,t) \geq \psi(x)$$



convective motion to the right



mask with small circle at origin

Constraints on Temporal Derivative

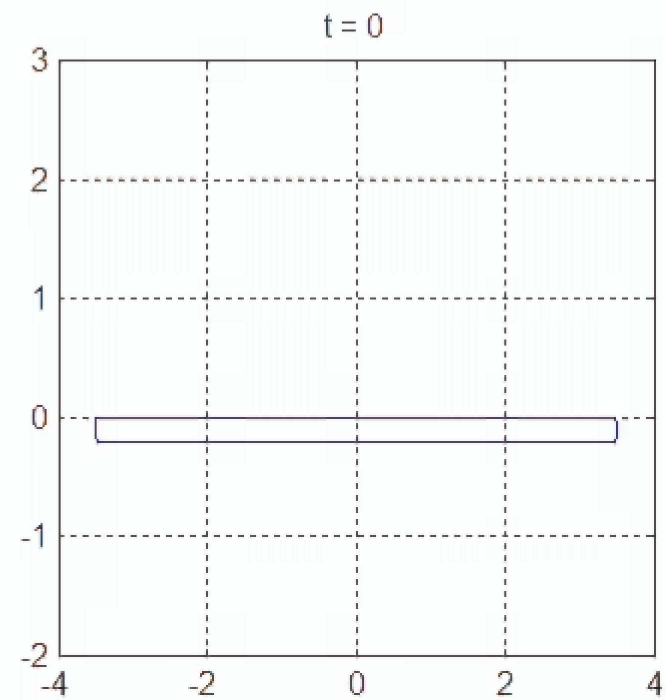
- Sign of temporal derivative controls whether implicit set can grow or shrink

$$D_t\phi(x,t) \leq 0 \qquad D_t\phi(x,t) \geq 0$$

- Example: reachable set only grows

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = W_p \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \frac{W_p}{R}\begin{bmatrix} y \\ -x \end{bmatrix} b + 2W_e \min\left(\sqrt{x^2+y^2}, S\right) a$$

$$a \in \mathbb{R}^2, \|a\| \leq 1$$
$$b \in [-1,+1]$$
$$W_p, W_e, R, S \text{ constant}$$



Stochastic Differential Equations (v1.1)

- Itô stochastic differential equation

$$dx(t) = f(x(t), t)dt + \sigma(x(t), t)dB(t)$$

- Kolmogorov or Fokker-Planck equation for expected outcome

$$D_t\phi + f^T \nabla \phi - \frac{1}{2}\,\mathrm{trace}\left[\sigma\sigma^T D_x^2 \phi\right] = 0$$

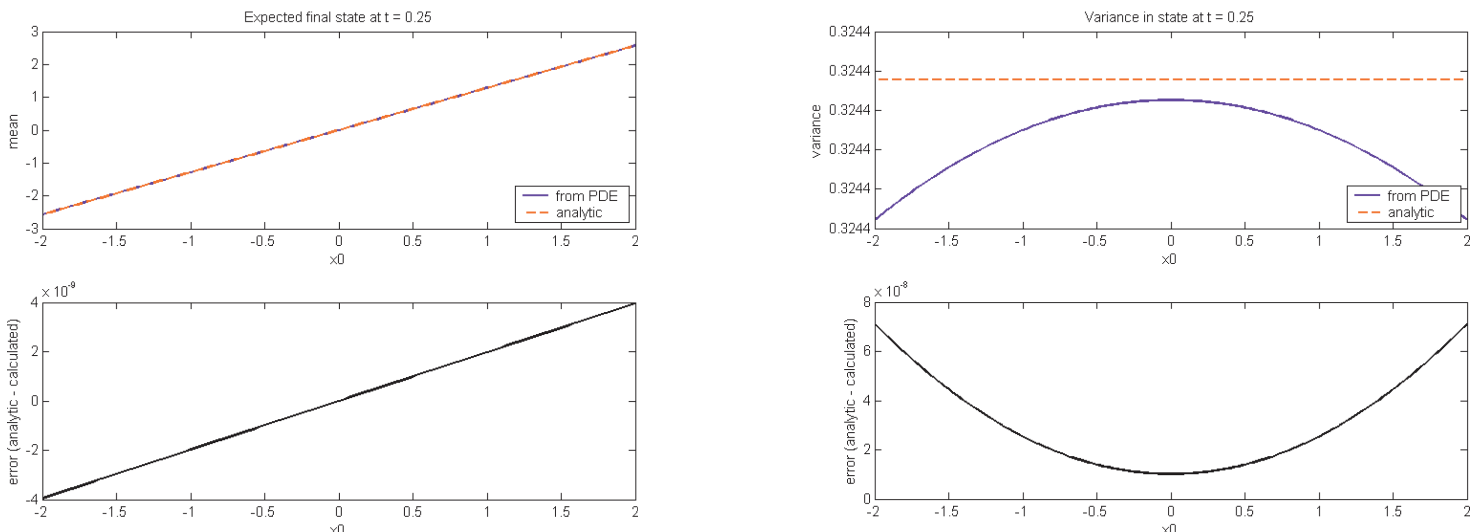
  – Example: linear DE with additive noise

$$f(x,t) = ax,\ \sigma(x,t) = b \text{ where } a = 1,\ b = 0.1$$

$t = 0.25$

value

error



final state mean

final state variance

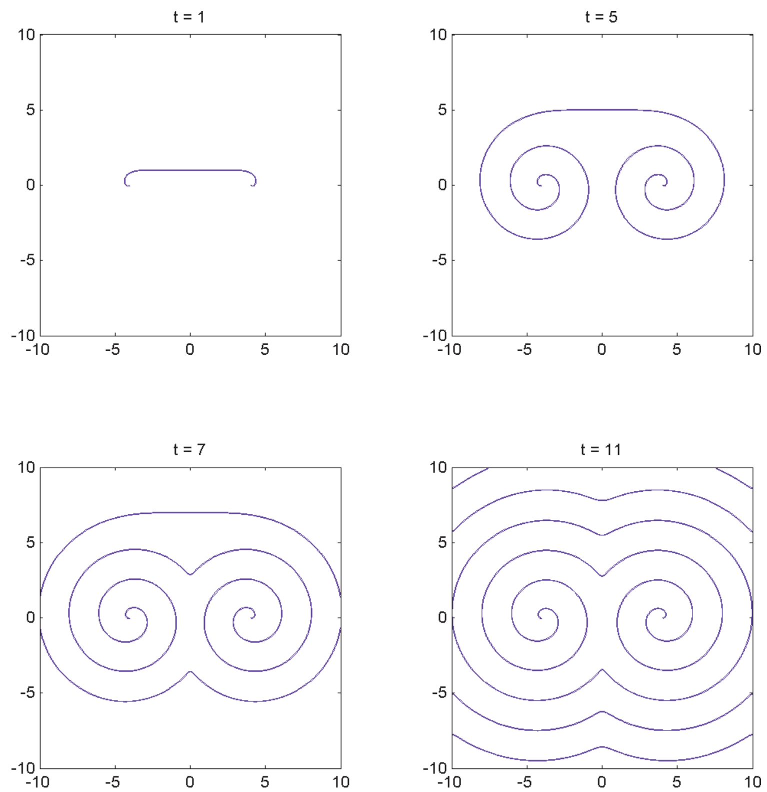
Open Curves by Vector Level Sets (v1.1)

- Normal level set methods can only represent closed curves
- Evolve two level sets in unison to represent an open curve Γ

$$D_t\phi - \text{sign}(\psi)\left[\lambda\,\text{sign}(\psi)\kappa(\phi) - 1\right]|\nabla\phi| = 0$$

$$D_t\psi - \text{sign}(\phi)\left[\lambda\,\text{sign}(\phi)\kappa(\psi) + 1\right]|\nabla\psi| = 0$$

$$\Gamma(t) = \{x \mid \phi(x,t) = 0 \wedge \psi(x,t) > 0\}$$



Continuous Reachable Sets

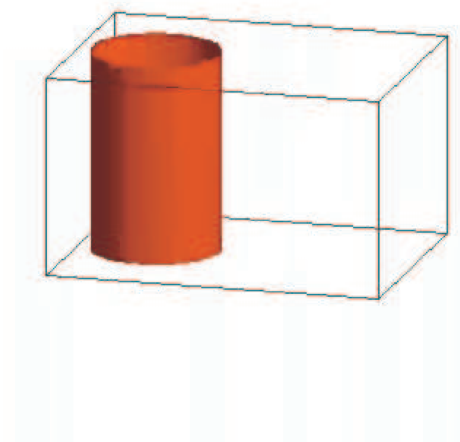
- Nonlinear dynamics with adversarial inputs

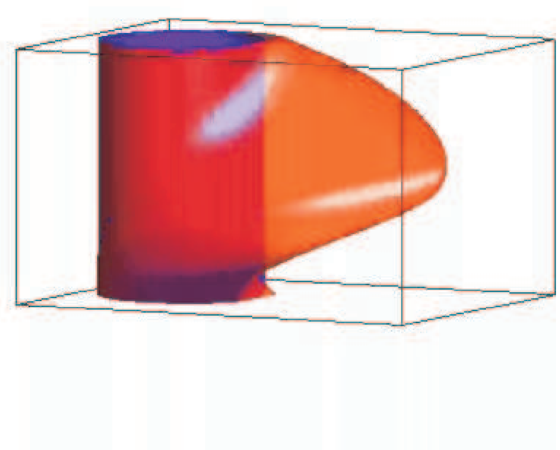
$$D_t\phi(x,t) + \min\left[0, H(x, \nabla\phi(x,t))\right] = 0$$

$$H(x,p) = \max_{a\in\mathcal{A}} \min_{b\in\mathcal{B}} \left[p \cdot f(x,a,b)\right]$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -v_a + v_b \cos x_3 + a x_2 \\ v_b \sin x_3 - a x_1 \\ b - a \end{bmatrix}$$
$$= f(x,a,b)$$

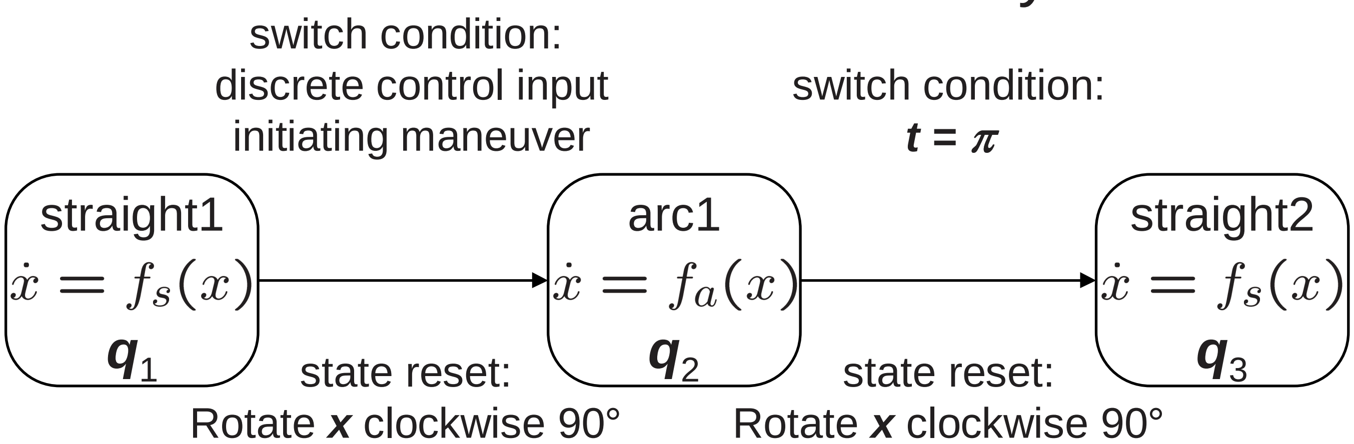
$$a \in \mathcal{A} = [-1, +1]$$
$$b \in \mathcal{B} = [-1, +1]$$
$$v_a, v_b \text{ constant}$$





Hybrid System Reachable Sets

- Mixture of continuous and discrete dynamics

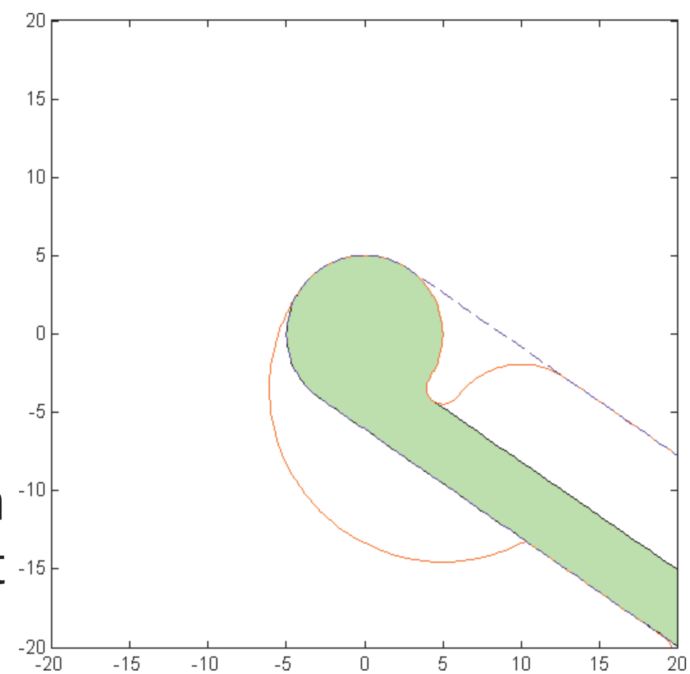


$$f_s(x) = \begin{bmatrix} -v_a + v_b \cos \psi_r \\ v_a \sin \psi_r \end{bmatrix}$$

$$f_a(x) = \begin{bmatrix} -v_a + v_b \cos \psi_r + \omega x_2 \\ v_a \sin \psi_r - \omega x_1 \end{bmatrix}$$

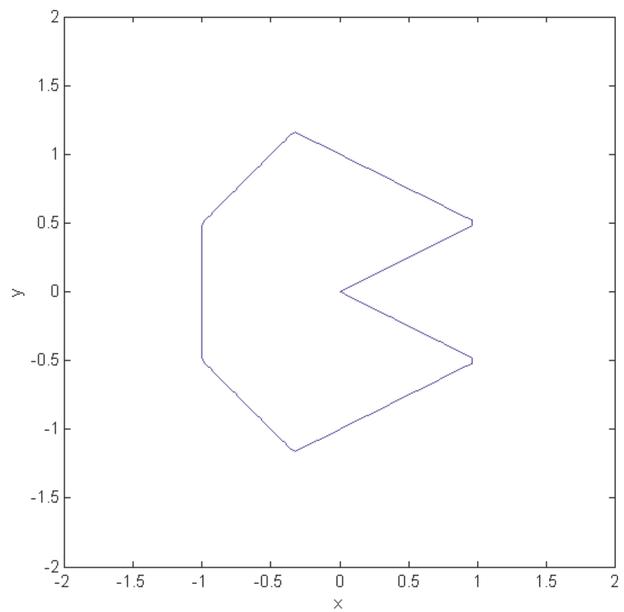
v_a, v_b, ψ_r, ω constant

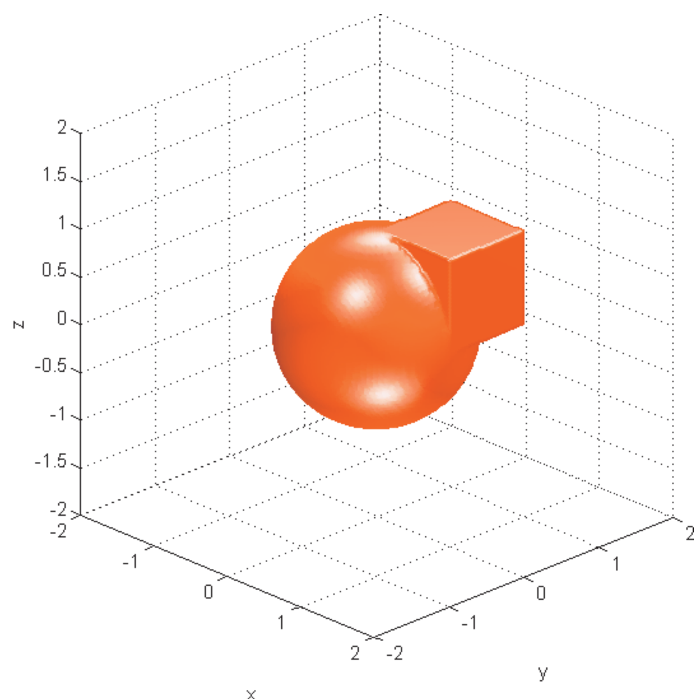
set of states leading to collision
whether maneuver is initiated or not



Constructive Solid Geometry

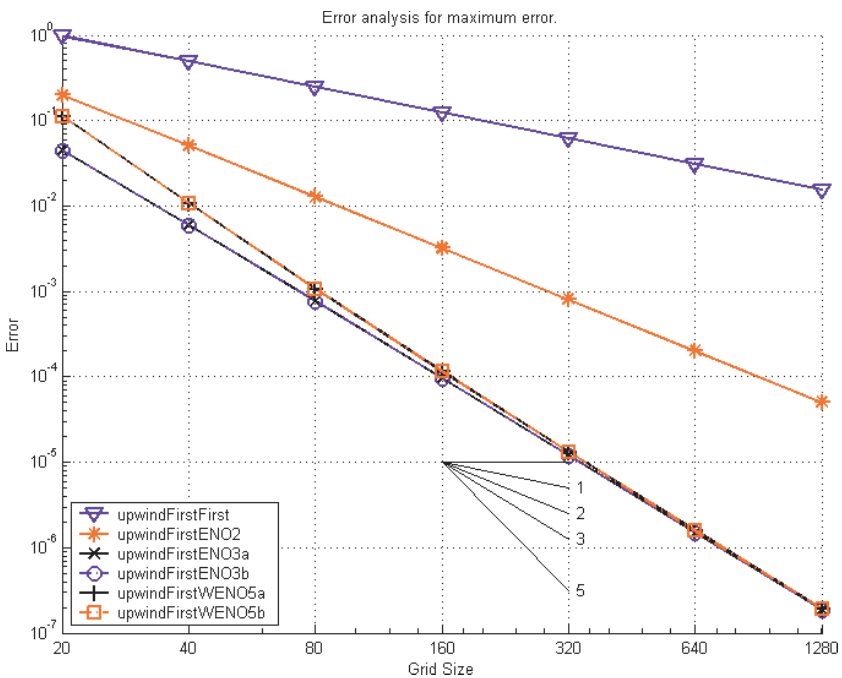
- Simple geometric shapes have simple algebraic implicit surface functions
  - Circles, spheres, cylinders, hyperplanes, rectangles
- Simple set operations correspond to simple mathematical operations on implicit surface functions
  - Intersection, union, complement, set difference



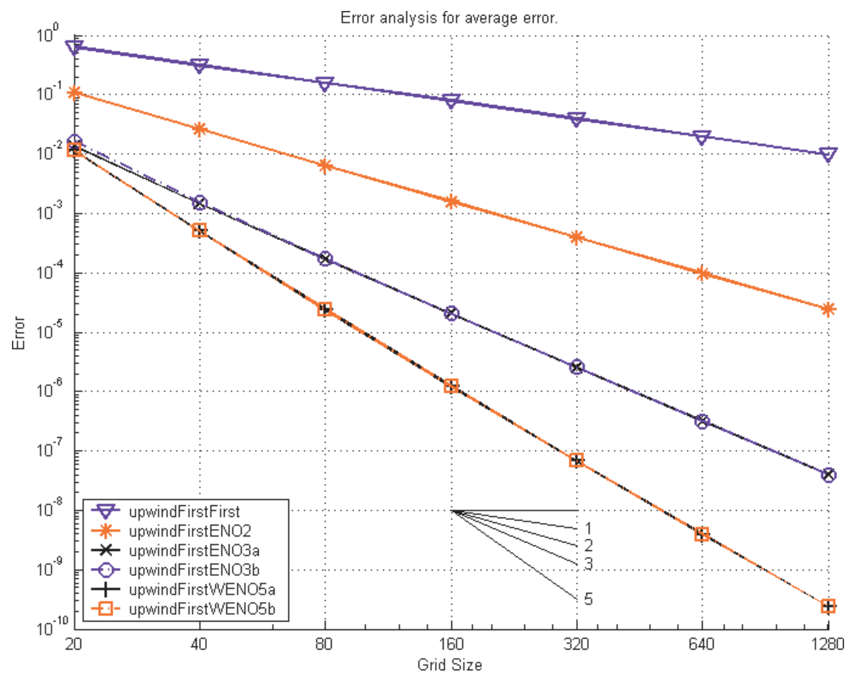


High Order Accuracy

- Temporally: explicit, Total Variation Diminishing Runge-Kutta integrators of order one to three
- Spatially: (Weighted) Essentially Non-Oscillatory upwind finite difference schemes of order one to five
  - Example: approximate derivative of function with kinks



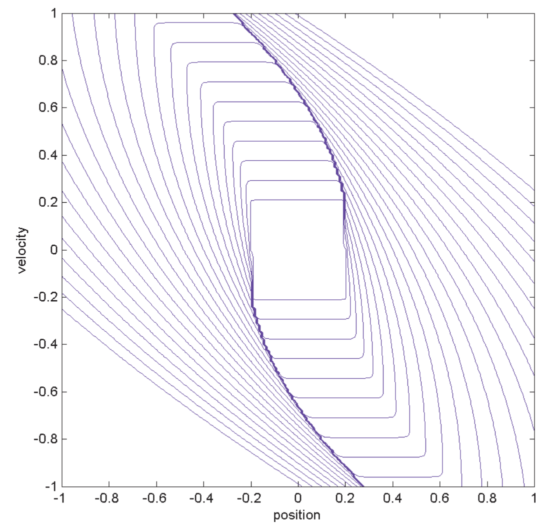
maximum error

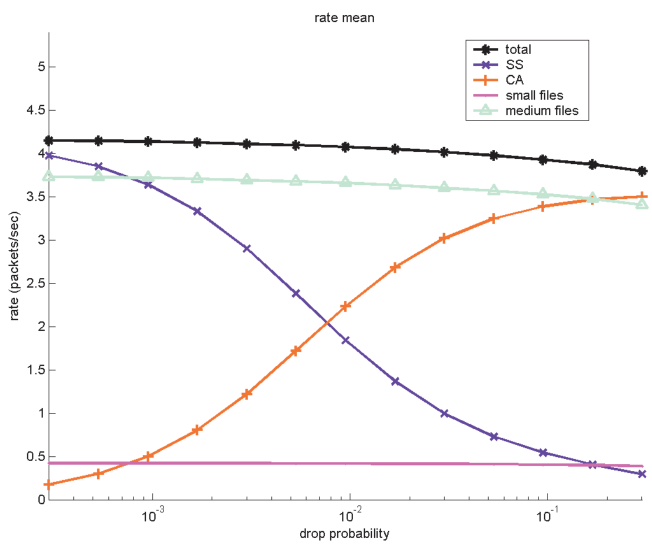


average error

Other Available Examples

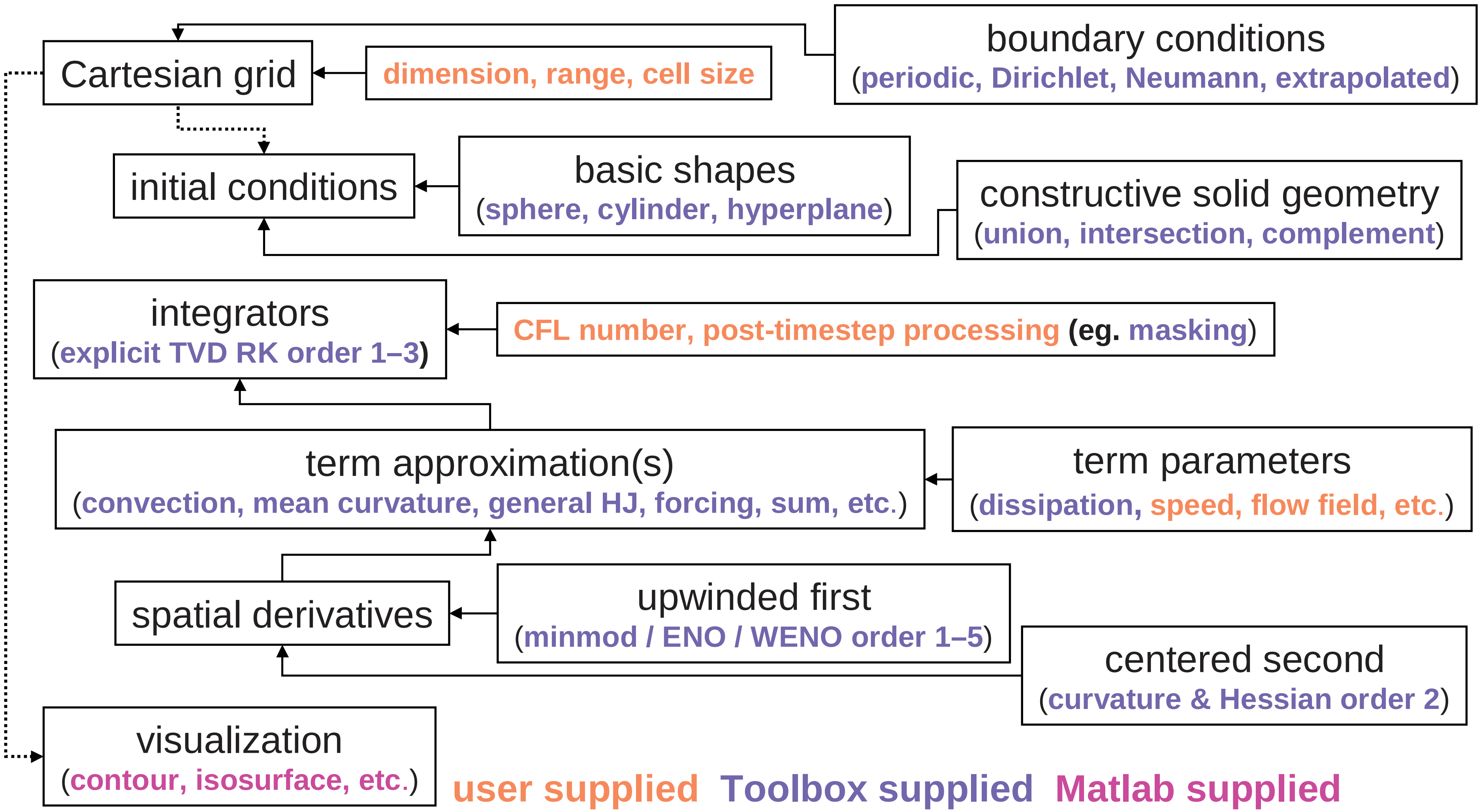
- Hybrid Systems Computation & Control
  - Mitchell & Templeton (2005)
  - Stationary HJ PDE for minimum time to reach or cost to go
  - Stochastic hybrid system model of Internet TCP transmission rate
- Journal of Optimization Theory & Applications
  - Kurzhanski, Mitchell & Varaiya (to appear 2006)
  - State constrained optimal control





The Toolbox: How to Use It

- Cut and paste from existing examples
- Most code is for initialization and visualization

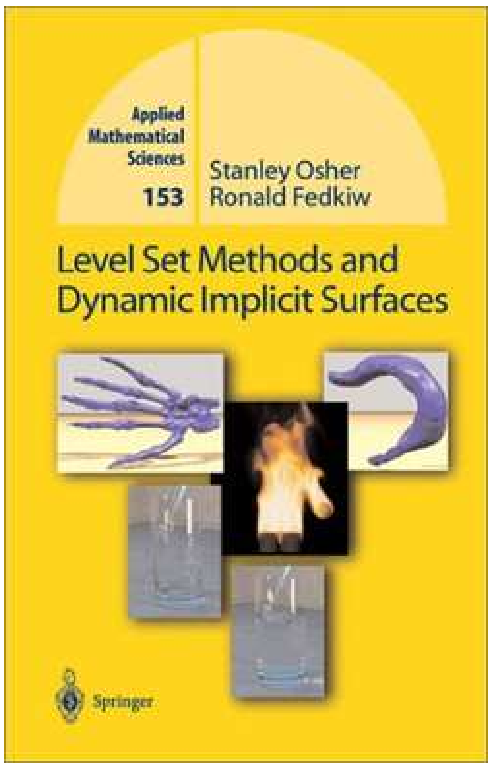


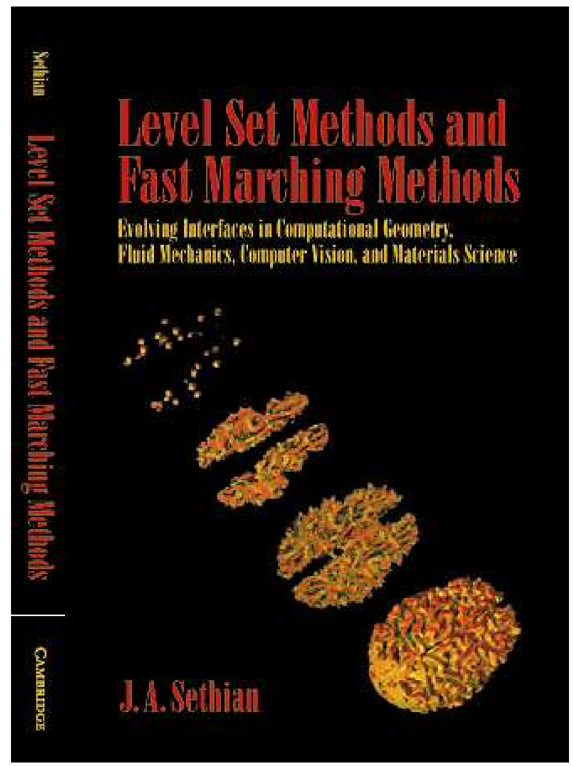
Future Work

- Algorithms
  - Implicit temporal integrators
  - Fast methods for stationary Hamilton-Jacobi
  - General boundary conditions
  - Other numerical Hamiltonians
  - Monotone schemes for second derivatives
  - ENO / WENO function value interpolation
  - Particle level set methods
  - Adaptive grids
- More application examples
  - Surfaces of codimension two
  - Hybrid system reachable sets and verification
  - Path planning for robotics
  - Image processing, financial math, fluid dynamics, etc.

The Toolbox is not a Tutorial

- Users will need to read the literature
- Two textbooks are available
  - Osher & Fedkiw (2002)
  - Sethian (1999)





The Toolbox: Why Use It?

- Dynamic implicit surfaces and Hamilton-Jacobi equations have many practical applications
- The toolbox provides an environment for exploring and experimenting with level set methods
  - Fourteen examples
  - Approximations of most common types of motion
  - High order accuracy
  - Arbitrary dimension
  - Reasonable speed with vectorized code
  - Direct access to Matlab debugging and visualization
  - Source code for all toolbox routines
- The toolbox is free for research use

  `http://www.cs.ubc.ca/~mitchell/ToolboxLS`

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For more information contact

Ian Mitchell

Department of Computer Science

The University of British Columbia

mitchell@cs.ubc.ca

http://www.cs.ubc.ca/~mitchell