A Toolbox of Hamilton-Jacobi Solvers for Analysis of Nondeterministic Continuous and Hybrid Systems

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Nondeterministic, Nonlinear Systems

$$\dot{x} = f(x, p)$$

- Systems with unknown parameters $p(t)$
- Bounded value inputs $p(t) \in P$
  - Controls: double integrator time to reach
  - Disturbances: robust reach sets
- Stochastic perturbations $p(t) \sim P$
  - Continuous state Brownian motion: double integrator with stochastic viscosity
  - Discrete state Poisson processes: stochastic hybrid system model of TCP communication protocol

Hamilton-Jacobi Equations

$$D_t\varphi(x,t) + G(x,t,\varphi,\nabla\varphi,D_x^2\varphi) = 0$$

$$\varphi(x,0) = g(x) \text{ bounded and continous}$$

$$G(x,t,r,p,\mathbf{X}) \leq G(x,t,s,p,\mathbf{Y}), \text{ if } r \leq s \text{ and } \mathbf{Y} \leq \mathbf{X}$$

- Time-dependent partial differential equation (PDE)
- In general, classical solution will not exist
  - Viscosity solution φ will be continuous but not differentiable
- For example, classical Hamilton-Jacobi-Bellman
  - Finite horizon optimal cost leads to terminal value PDE

$$\varphi(x(t),t) = \min_{u(\cdot)} \left[g(x(T)) + \int_t^T \ell(x(s),u(s))ds \right]$$

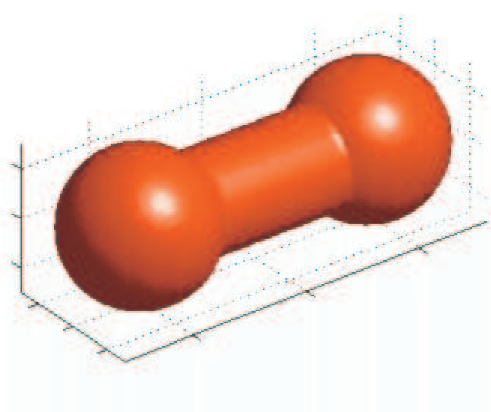
$$D_t\varphi(x,t) + \min_u \left[\nabla\varphi(x,t)\cdot f(x,u) + \ell(x,u)\right] = 0$$

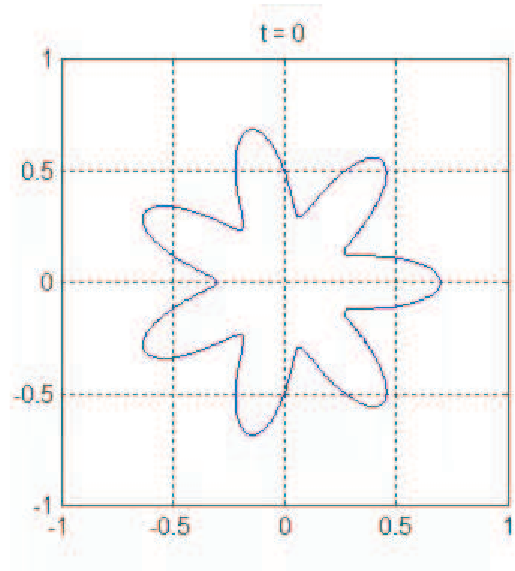
The Toolbox of Level Set Methods

- A collection of Matlab routines to approximate the viscosity solution of time-dependent HJ PDEs
  - Fixed Cartesian grids
  - Arbitrary dimension (computational resource limited)
  - Vectorized code achieves reasonable speed
  - Direct access to Matlab debugging and visualization
  - Source code is provided for all toolbox routines
- Underlying algorithms
  - Solve various forms of Hamilton-Jacobi PDE
  - First and second spatial derivatives
  - First temporal derivatives
  - High order accurate approximation schemes
  - Explicit temporal integration

Level Set Methods

- Numerical algorithms for dynamic implicit surfaces and Hamilton-Jacobi partial differential equations
- Applications in
  - Graphics, Computational Geometry and Mesh Generation
  - Financial Mathematics and Stochastic Differential Equations
  - Fluid and Combustion Simulation
  - Image Processing and Computer Vision
  - Robotics, Control and Dynamic Programming
  - Verification and Reachable Sets





Why Use It?

- Does not escape Bellman's curse of dimensionality
  - Dimensions 1–3 interactively, 3–5 slow but feasible
- Pedagogical tool
  - Experiment with optimal control and differential game problems that have no analytic solution
  - Access to Matlab's visualization & debugging
  - Source code for all routines and examples
  - Reasonable speed with vectorized code
- Validation of faster but more specialized algorithms
  - Reduced order TCP model assumed form of high order moments of the distribution
- Study low dimensional systems
  - Mobile robots in 2–3 spatial dimensions
- Free (google "toolbox level set methods")

Using the Toolbox

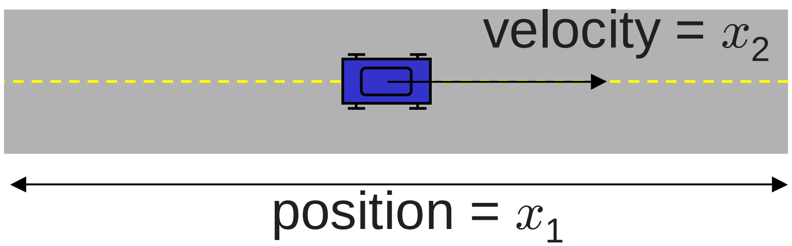
- Similar to Matlab's ODE integrators
  - More parameters to specify
  - Formulation and scaling must be considered
  - Many examples are available
- PDE forms applicable to systems analysis

$$
\begin{aligned}
0 = & D_t \varphi(x,t) \\
& + v(x,t) \cdot \nabla \varphi(x,t) \\
& + H(x,t,\varphi,\nabla\varphi) \\
& - \text{trace}[\mathbf{L}(x,t) D_x^2 \varphi(x,t) \mathbf{R}(x,t)] \\
& + \lambda(x,t)\varphi(x,t) \\
& + F(x,t,\varphi),
\end{aligned}
$$

$$
\begin{aligned}
D_t\varphi(x,t) &\geq 0, & \qquad D_t\varphi(x,t) &\leq 0, \\
\varphi(x,t) &\leq \psi(x,t), & \qquad \varphi(x,t) &\geq \psi(x,t),
\end{aligned}
$$

Example: Optimal Cost to Go

- Specifically, study the classical double integrator
  - Bring point-like dynamic vehicle to a halt at the origin in minimum time, subject to acceleration bound $|b| \leq 1$
- Leads to stationary (time-independent) HJ PDE



$$\dot{x} = \begin{bmatrix} x_2 \\ b \end{bmatrix}, \qquad |b| \leq 1$$

Stationary Hamilton-Jacobi

General cost to go function

$$\vartheta(x) = \inf_{b(\cdot)} \int_0^T \ell(x(t), b(t))dt,$$

for closed target set $\mathcal{T}$,
continuous running cost $\ell(x, b) > 0$,
and terminal time

$$T = \min\{t \geq 0 \mid x(t) \in \mathcal{T}\}.$$

If $\ell \equiv 1$, then $\vartheta(x)$ is the minimum time to reach $\mathcal{T}$.

To solve, find viscosity solution of

$$\min_{b \in \mathcal{B}} [\nabla \vartheta(x) \cdot f(x, b) - \ell(x, b)] = 0 \text{ in } \mathbb{R}^d \setminus \mathcal{T},$$

$$\vartheta(x) = 0 \text{ on } \partial\mathcal{T}.$$

Transformation to Time-Dependent HJ

Create implicit surface definition of $\mathcal{T}$

$$\varphi(x,0) \begin{cases} \leq 0, x \in \mathcal{T}; \\ = 0, x \in \partial\mathcal{T}; \\ \geq 0, x \in \mathbb{R}^d \setminus \mathcal{T}. \end{cases}$$

Under assumption $\nabla\varphi(x,0) \cdot f(x,b) \neq 0$ on $\partial\mathcal{T}$, make change of variables

$$\nabla\vartheta(x) \leftarrow \frac{\nabla\varphi(x,t)}{D_t\varphi(x,t)}$$

and get toolbox appropriate PDE

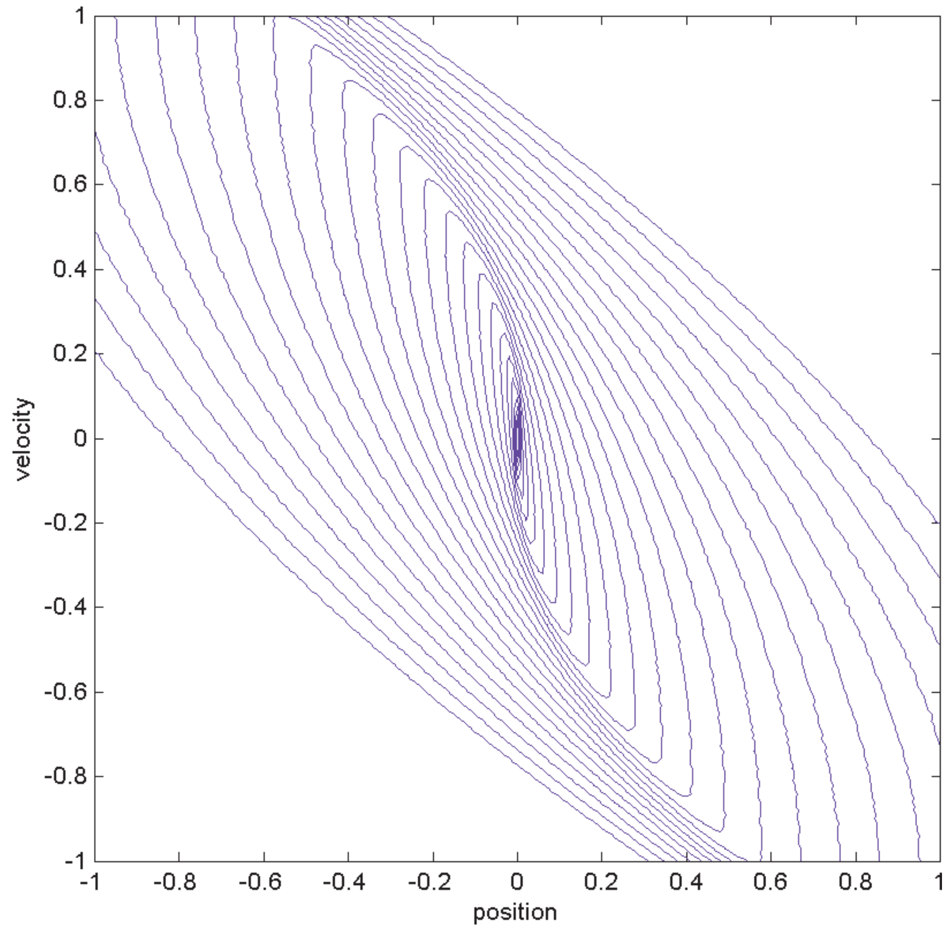
$$D_t\varphi(x,t) + \min_{b\in\mathcal{B}} \frac{\nabla\varphi(x,t)\cdot f(x,b)}{\ell(x,b)} = 0.$$

After solving, set ϑ to be crossing time

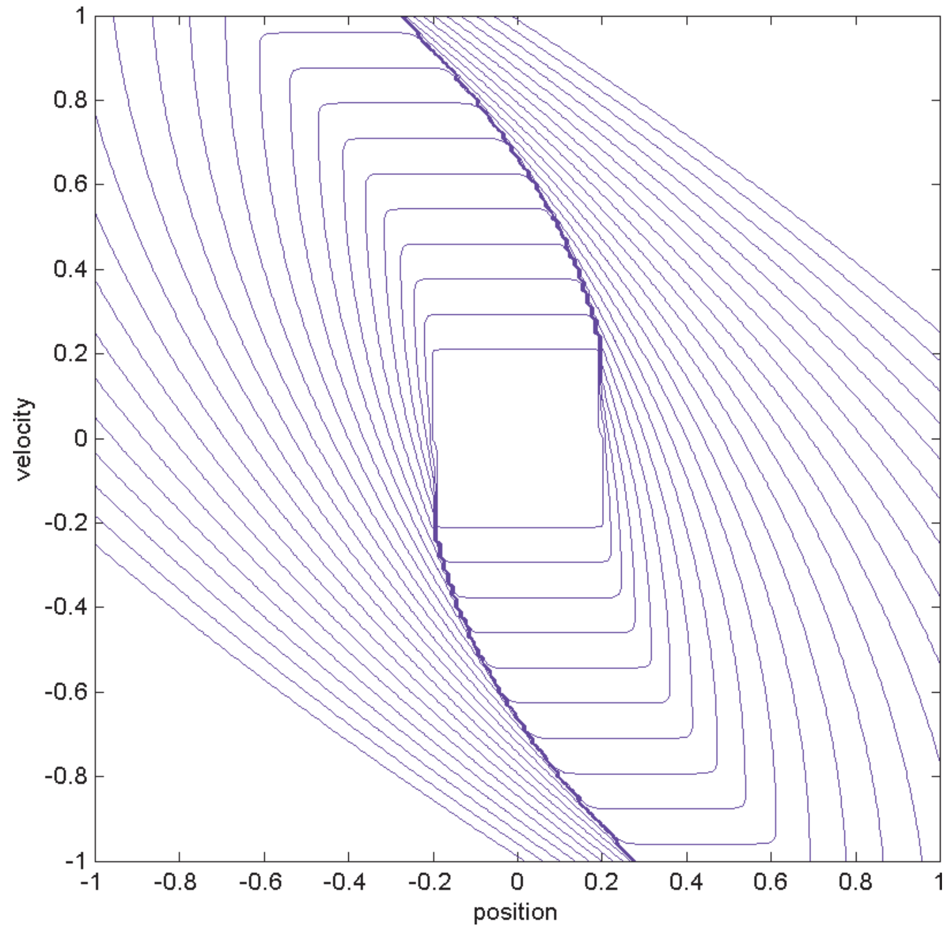
$$\vartheta(x) = \{t \mid \varphi(x,t) = 0\}.$$

Double Integrator Time to Reach

- Contours of minimum time to reach $\vartheta(x)$



Target Radius 0



Target Radius 0.2

Implemented in the Toolbox

- Part of the standard toolbox distribution (version 1.1 beta)
  - **`Examples/TimeToReach/doubleIntegratorTTR`**
- PDE terms utilized

$$
\begin{aligned}
0 = & D_t\varphi(x,t) \\
& + v(x,t)\cdot\nabla\varphi(x,t) \\
& + H(x,t,\varphi,\nabla\varphi) \\
& - \text{trace}[\mathbf{L}(x,t)D_x^2\varphi(x,t)\mathbf{R}(x,t)] \\
& + \lambda(x,t)\varphi(x,t) \\
& + F(x,t,\varphi),
\end{aligned}
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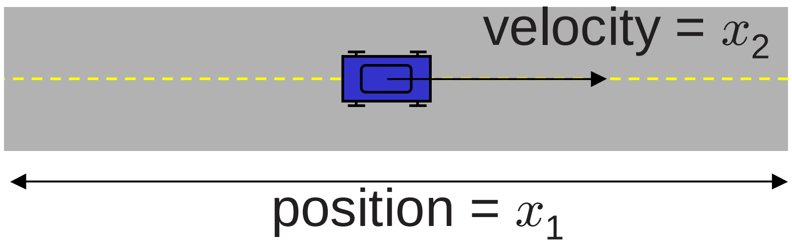
\end{aligned}
$$

Example: Stochastic Continuous System

- Underlying double integrator model
  - Stochastically varying wind friction (viscosity)
  - Minimize continuous terminal cost $g(x)$ at fixed finite time horizon



$$\dot{x} = \begin{bmatrix} x_2 \\ b - k_1 x_2 \end{bmatrix} dt - \begin{bmatrix} 0 \\ k_2 x_2 \end{bmatrix} dB(t),$$

$$|b| \leq 1 \text{ and } B(t) \text{ Brownian motion.}$$

$$g(x) = 1 - \left[1 + \exp\left(\frac{\|x\|_2 - \rho}{\epsilon \rho} \right) \right]^{-1},$$

Stochastic Differential Game

Expected cost with fixed horizon T

$$\varphi(x_0, t_0) = E\left[\inf_{b(\cdot)} \sup_{a(\cdot)} \left(\int_{t_0}^{T} \ell(x, s, a, b) ds + g(x(T))\right)\right].$$

where system evolves according to SDE

$$dx(t) = f(x(t), t, a, b)dt + \sigma(x(t), t)dB(t), \qquad x(t_0) = x_0,$$

with adversarial inputs a and b.

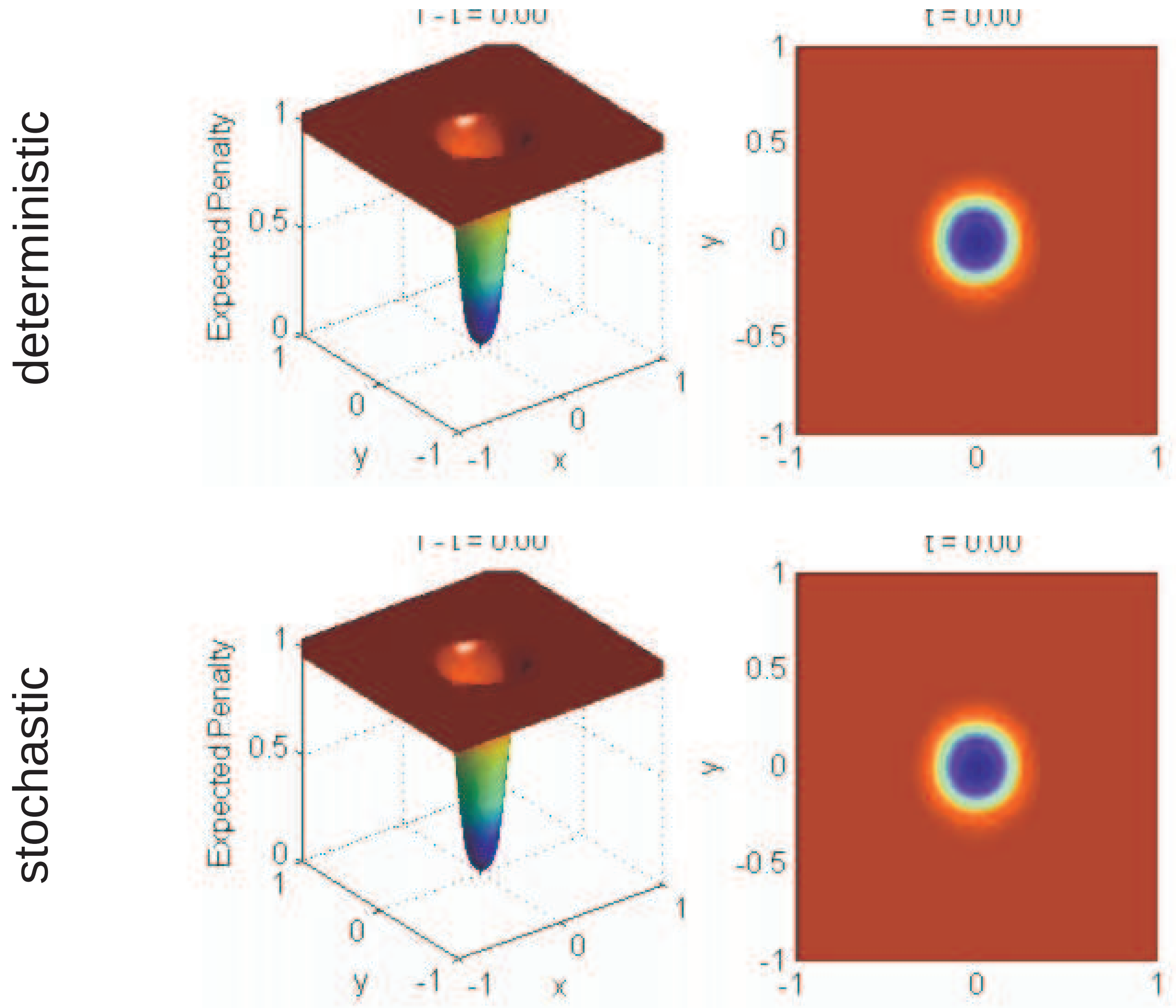
Find viscosity solution of

$$D_t\varphi + H(x, t, \nabla\varphi) + \tfrac{1}{2}\,\mathrm{trace}\left[\sigma\sigma^T D_x^2 \varphi\right] = 0.$$

$$H(x, t, p) = \max_{a \in \mathcal{A}} \min_{b \in \mathcal{B}} \left[p \cdot f(x, t, a, b) + \ell(x, t, a, b)\right],$$

$$\varphi(x, T) = g(x).$$

Stochastic Double Integrator Results



Implemented in the Toolbox

- Separate code release on toolbox website
  - **`PublicationCode/HSCC2005/SDE/viscousIntegrator`**
- PDE terms utilized

$$
\begin{aligned}
0 = & D_t\varphi(x,t) \\
& + v(x,t) \cdot \nabla\varphi(x,t) \\
& + H(x,t,\varphi,\nabla\varphi) \\
& - \text{trace}[\mathbf{L}(x,t) D_x^2 \varphi(x,t) \mathbf{R}(x,t)] \\
& + \lambda(x,t)\varphi(x,t) \\
& + F(x,t,\varphi),
\end{aligned}
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\begin{aligned}
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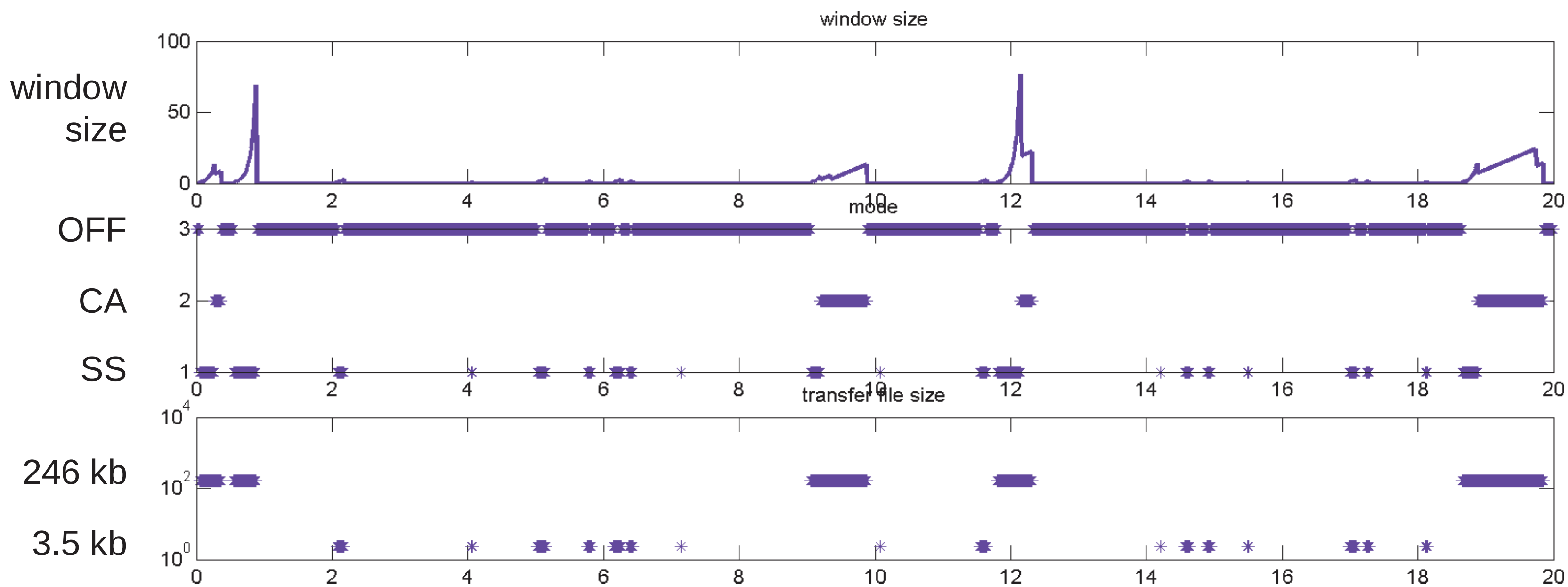
\varphi(x,t) \leq \psi(x,t), \qquad\qquad & \varphi(x,t) \geq \psi(x,t),
\end{aligned}
$$

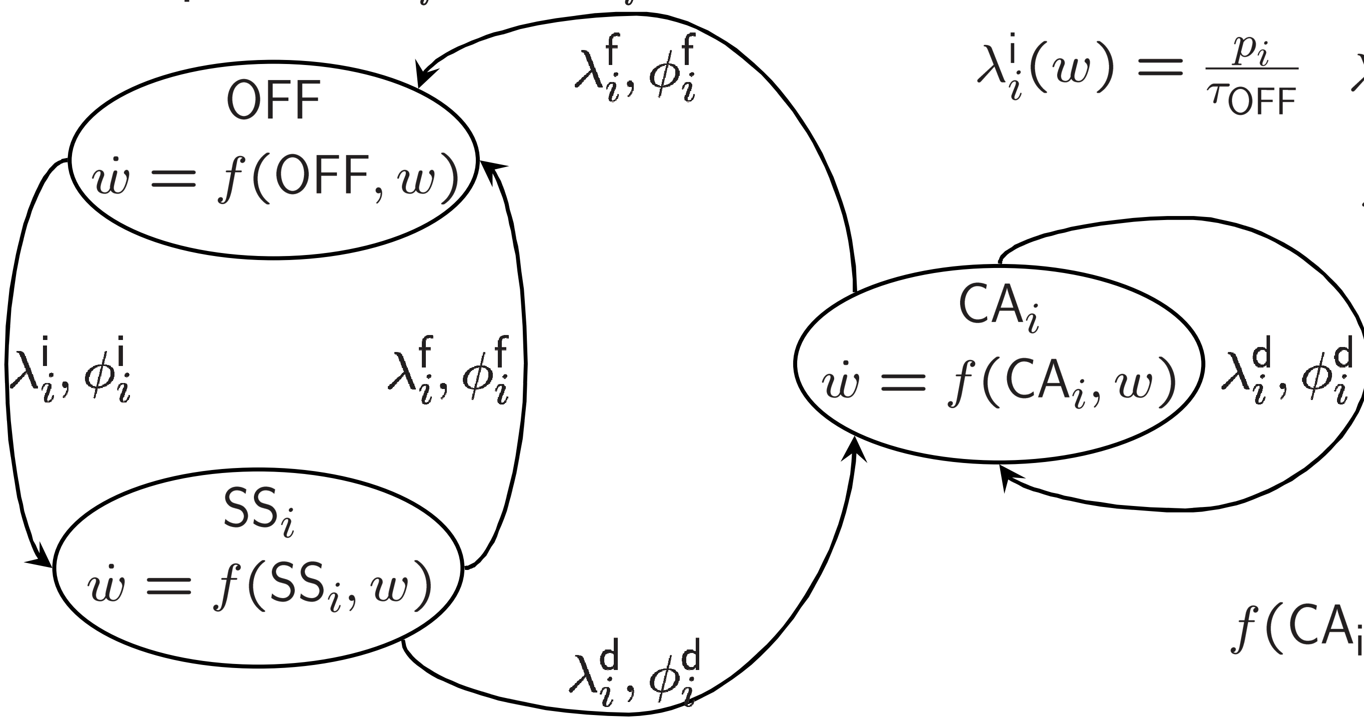
Transmission Control Protocol (TCP)

- Handles reliable end-to-end delivery of packets over Internet
  - Window size w controls transmission rate
  - Permitted number of transmitted but unacknowledged packets
- When transmitting a file, connection is in one of two states:
  - Slow Start (SS): window size grows exponentially
  - Congestion Avoidance (CA): window size grows linearly
- When a packet is dropped:
  - Switch to CA and cut window size in half



Example: Stochastic Hybrid System

- Window size is continuous variable, evolves deterministically
- Discrete transitions
  - Start of transfer, packet drop, end of transfer
  - Occur at “instantaneous rate” λ, cause window size reset ϕ
- Separate SS_i and CA_i modes and transitions for each file size k_i



$$\lambda_i^{\mathsf{i}}(w) = \frac{p_i}{\tau_{\mathsf{OFF}}} \quad \lambda_i^{\mathsf{d}}(w) = \frac{p_{\mathsf{drop}} w}{\mathsf{RTT}}$$

$$\lambda_i^{\mathsf{f}}(w) = \frac{w}{k_i \mathsf{RTT}}$$

$$f(\mathsf{CA_i}, w) = \frac{1}{n_{\mathsf{ack}} \mathsf{RTT}}$$

[Hespanha, HSCC 2004]

$$f(\mathsf{OFF}, w) = 0 \quad f(\mathsf{SS_i}, w) = \frac{(\log 2) w}{n_{\mathsf{ack}} \mathsf{RTT}}$$

Stochastic Hybrid System

Terminal payoff

$$\varphi(q_0, x_0, t_0) = E[\varphi_T(q(T), x(T))],$$

where $q(t_0) = q_0$, $x(t_0) = x_0$, and
continuous evolution $\dot{x} = f(q, x, t)$.
Discrete transition or reset maps $(q, x) = \phi_j(q^-, x^-, t)$
occuring at intensities $\lambda_j(q, x, t) \geq 0$
with $x \in \mathbb{R}^d$, $q \in \mathcal{Q}$, $j \in \{1, 2, \ldots, m\}$.

Then (for identity resets in x),
find viscosity solution of

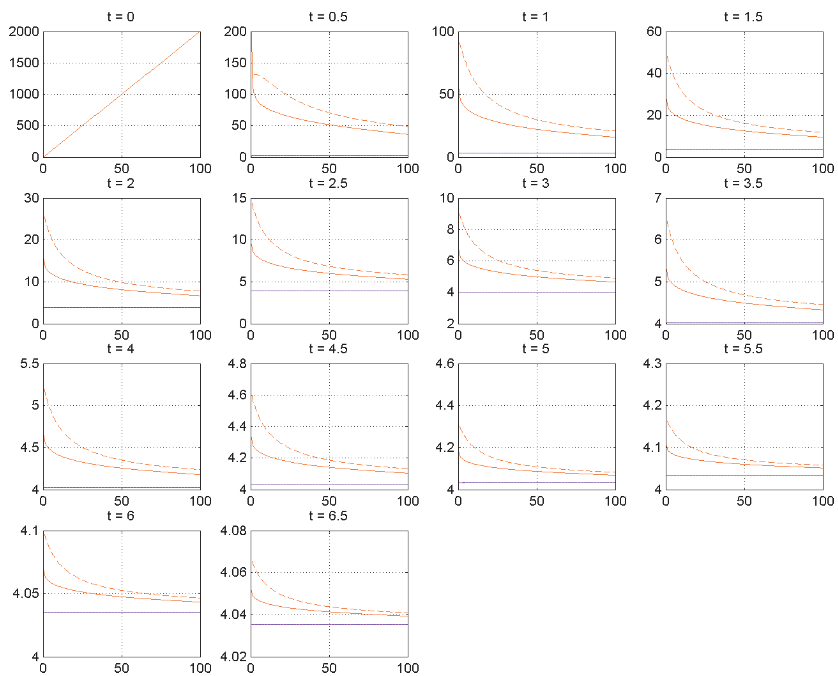
$$D_t\varphi(q, x, t) + \nabla\varphi(q, x, t) \cdot f(q, x, t) + \sum_{j=1}^{m} \lambda_j(q, x, t) \Big(\varphi(\phi_j(q, x, t), t) - \varphi(q, x, t) \Big) = 0$$

Steady State Measures of Rate

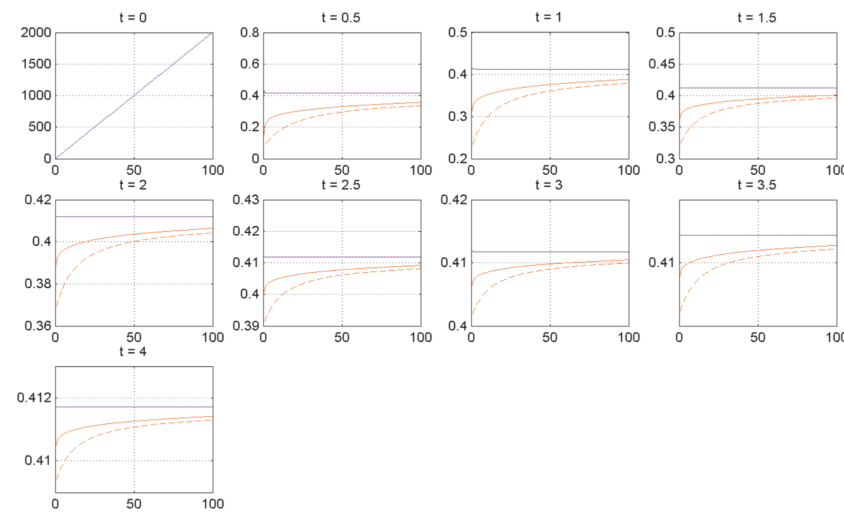
- Seek measures of rate = (window size / round trip time)
- For example, to find average rate over a set of modes $\mathcal{Q}_m$, solve PDE backwards in time to steady state with terminal conditions

$$\varphi_T(q,w) = \begin{cases} w, & \text{for } q \in \mathcal{Q}_m; \\ 0, & \text{otherwise.} \end{cases}$$

Mean Rate, All Modes

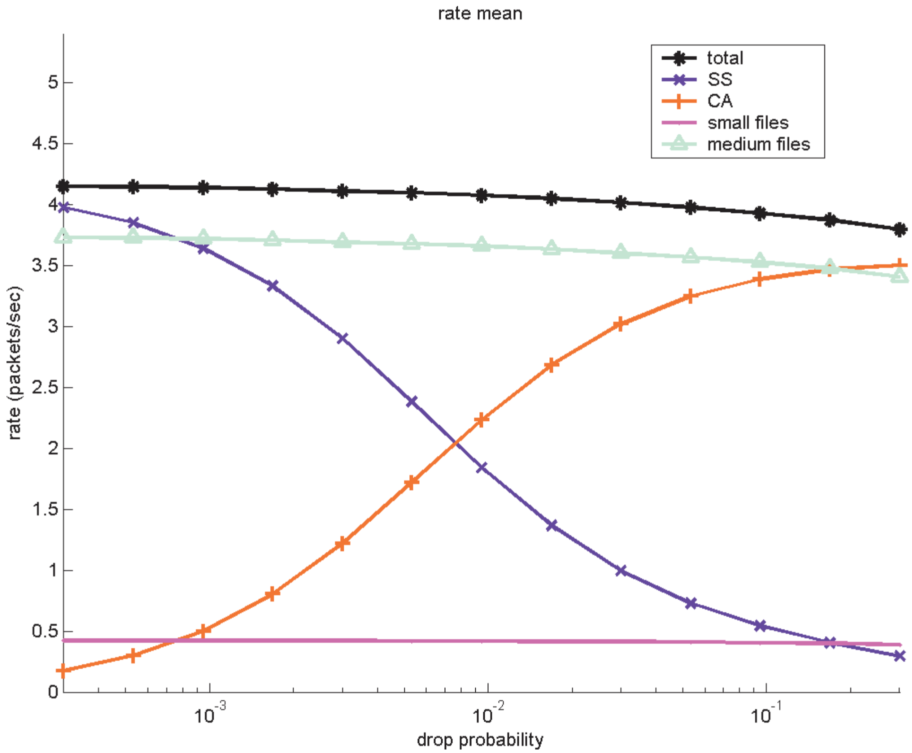


Mean Rate, Small Files Only

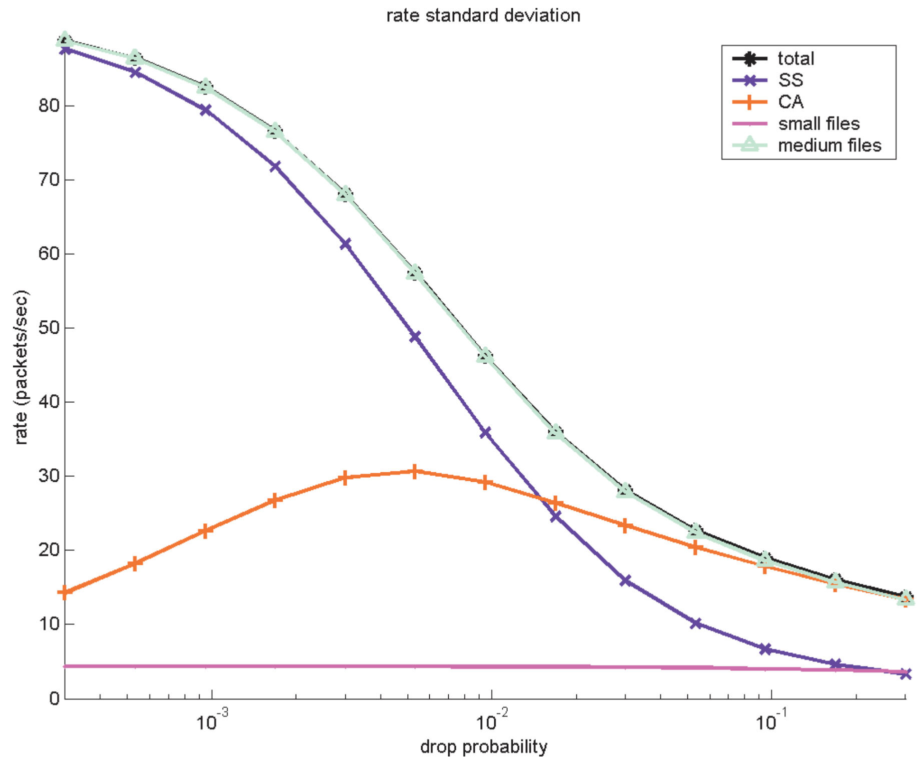


Measures of Rate Results

- Compare measures of rate for various drop probabilities
- Results match well with reduced order model
  - Validates assumption regarding high order distribution moments
  - [Hespanha, HSCC 2004] & [Hespanha, sub. Int. J. Hybrid Systems]



Rate Mean



Rate Standard Deviation

Implemented in the Toolbox

- Separate code release on toolbox website
  - **`PublicationCode/HSCC2005/CommunicationTCP/kolmogorovTCP`**
- PDE terms utilized

$$
\begin{aligned}
0 = & D_t\varphi(x,t) \\
& + v(x,t)\cdot\nabla\varphi(x,t) \\
& + H(x,t,\varphi,\nabla\varphi) \\
& - \mathrm{trace}[\mathbf{L}(x,t)D_x^2\varphi(x,t)\mathbf{R}(x,t)] \\
& + \lambda(x,t)\varphi(x,t) \\
& + F(x,t,\varphi),
\end{aligned}
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$$
\begin{aligned}
D_t\varphi(x,t) &\geq 0, & \qquad D_t\varphi(x,t) &\leq 0, \\
\varphi(x,t) &\leq \psi(x,t), & \qquad \varphi(x,t) &\geq \psi(x,t),
\end{aligned}
$$

Example: Continuous Reachable Sets

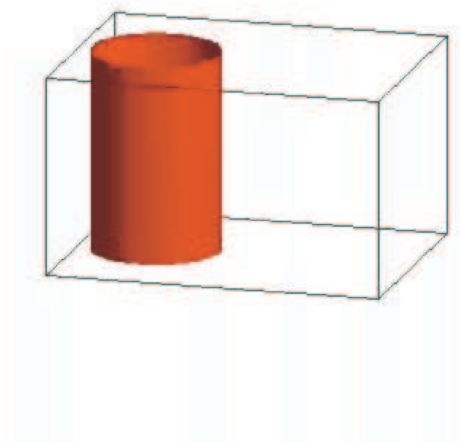
- Nonlinear dynamics with adversarial inputs

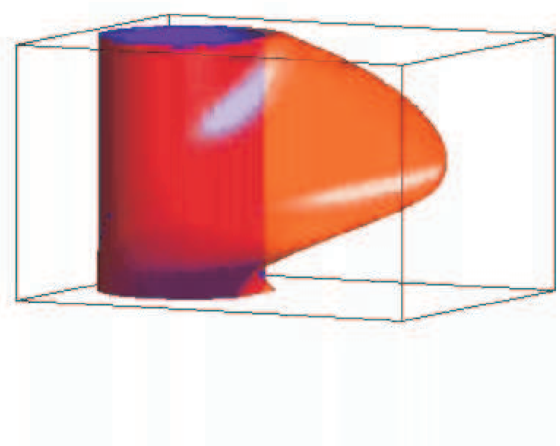
$$D_t\phi(x,t) + \min\left[0, H(x, \nabla\phi(x,t))\right] = 0$$

$$H(x,p) = \max_{a \in \mathcal{A}} \min_{b \in \mathcal{B}} \left[p \cdot f(x,a,b)\right]$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -v_a + v_b \cos x_3 + a x_2 \\ v_b \sin x_3 - a x_1 \\ b - a \end{bmatrix} = f(x,a,b)$$

$$a \in \mathcal{A} = [-1, +1]$$
$$b \in \mathcal{B} = [-1, +1]$$
$$v_a, v_b \text{ constant}$$



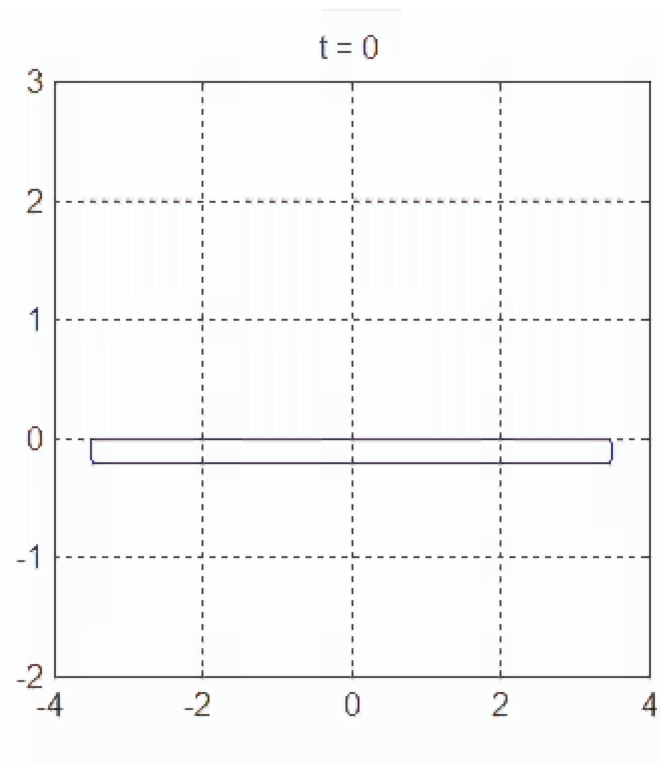


A Different Continuous Reachable Set

- Acoustic capture [Cardaliaguet, Quincampoix & Saint-Pierre, Ann. Int. Soc. Dynamic Games 1999]
  – Variation on homicidal chauffeur, where evader must reduce speed when near pursuer

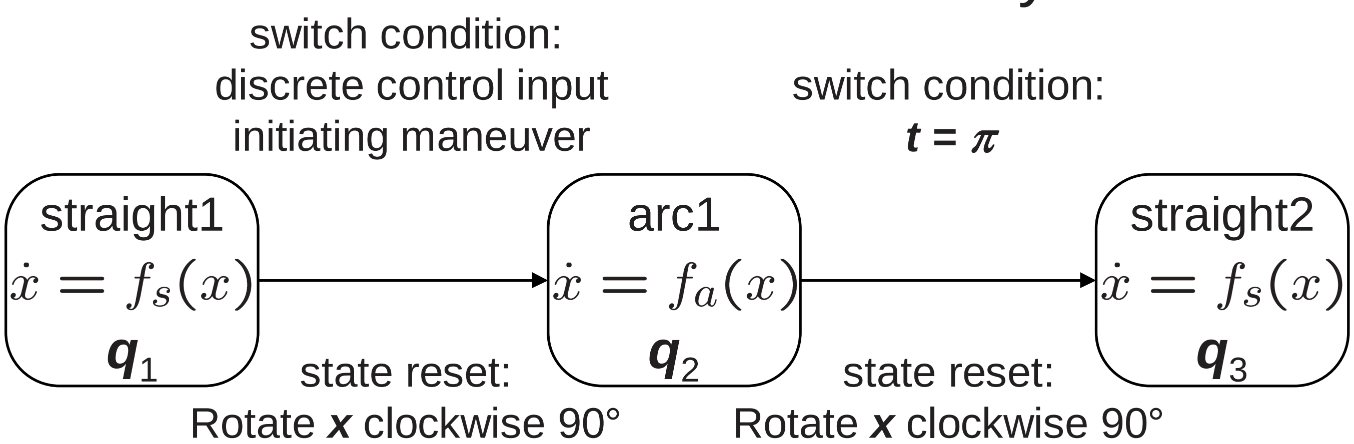
$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = W_p \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \frac{W_p}{R}\begin{bmatrix} y \\ -x \end{bmatrix} b + 2W_e \min\left(\sqrt{x^2+y^2}, S\right) a$$

$$a \in \mathbb{R}^2, \|a\| \leq 1$$
$$b \in [-1, +1]$$
$$W_p, W_e, R, S \text{ constant}$$



Example: Hybrid System Reachable Sets

- Mixture of continuous and discrete dynamics

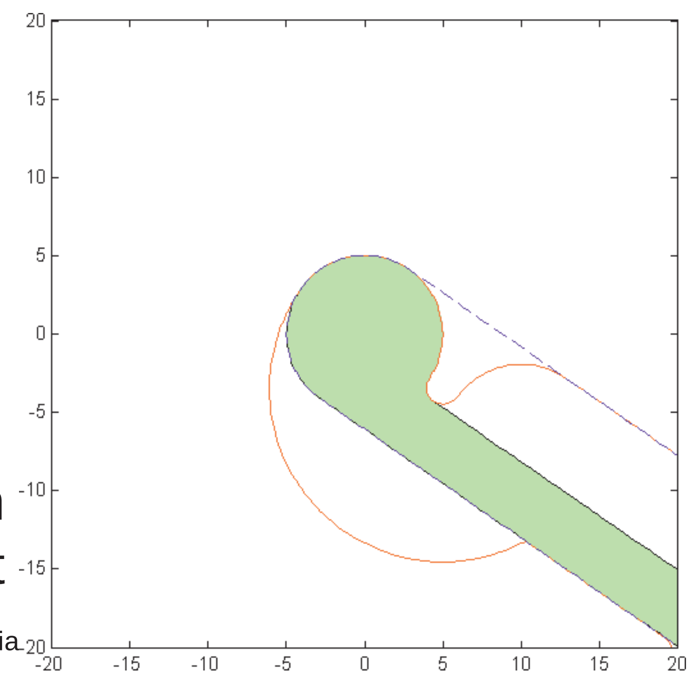


$$f_s(x) = \begin{bmatrix} -v_a + v_b \cos \psi_r \\ v_a \sin \psi_r \end{bmatrix}$$

$$f_a(x) = \begin{bmatrix} -v_a + v_b \cos \psi_r + \omega x_2 \\ v_a \sin \psi_r - \omega x_1 \end{bmatrix}$$

v_a, v_b, ψ_r, ω constant

set of states leading to collision
whether maneuver is initiated or not



Implemented in the Toolbox

- Part of the standard toolbox distribution (version 1.0)
  - **`Examples/Reachability/`**
- PDE terms utilized

$$
\begin{aligned}
0 = & D_t \varphi(x,t) \\
& + v(x,t) \cdot \nabla \varphi(x,t) \\
& + H(x,t,\varphi,\nabla\varphi) \\
& - \text{trace}[\mathbf{L}(x,t) D_x^2 \varphi(x,t) \mathbf{R}(x,t)] \\
& + \lambda(x,t)\varphi(x,t) \\
& + F(x,t,\varphi),
\end{aligned}
$$

$$
\begin{aligned}
D_t \varphi(x,t) &\geq 0, & \qquad D_t \varphi(x,t) &\leq 0, \\
\varphi(x,t) &\leq \psi(x,t), & \qquad \varphi(x,t) &\geq \psi(x,t),
\end{aligned}
$$

Future Work

- Toolbox additions
  - Implicit temporal integrators
  - Fast stationary Hamilton-Jacobi solvers
  - Particle level set methods
  - Adaptive grids
- More application examples
  - Hybrid system reachable sets
  - Image processing
  - Financial instrument pricing
- Wish List
  - Full nondeterministic hybrid system theory
  - Toolbox front end for specifying hybrid system verification problems—requires (nondeterministic) hybrid system specification language

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For more information contact

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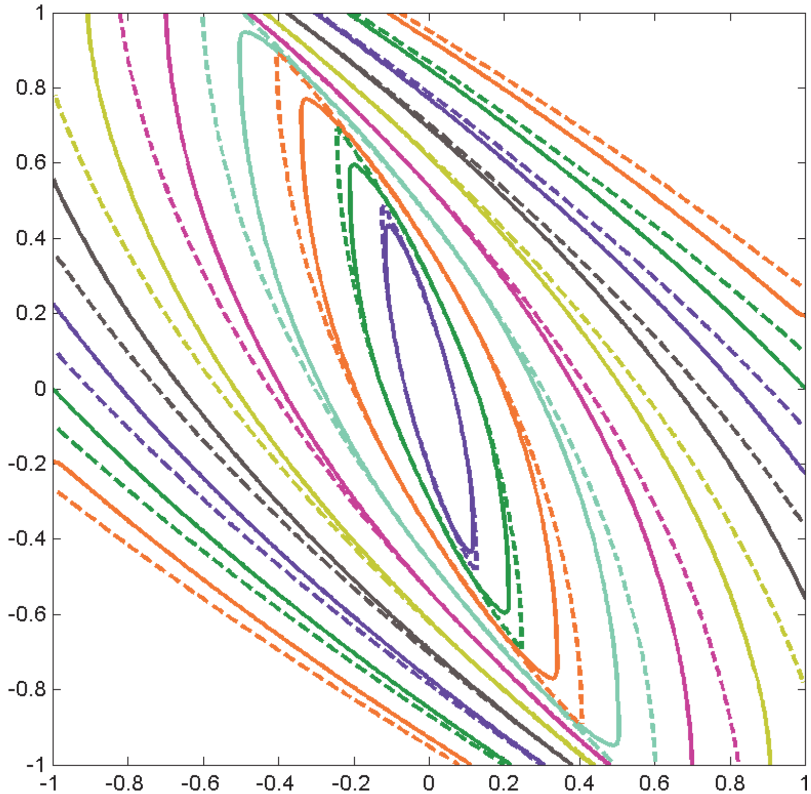
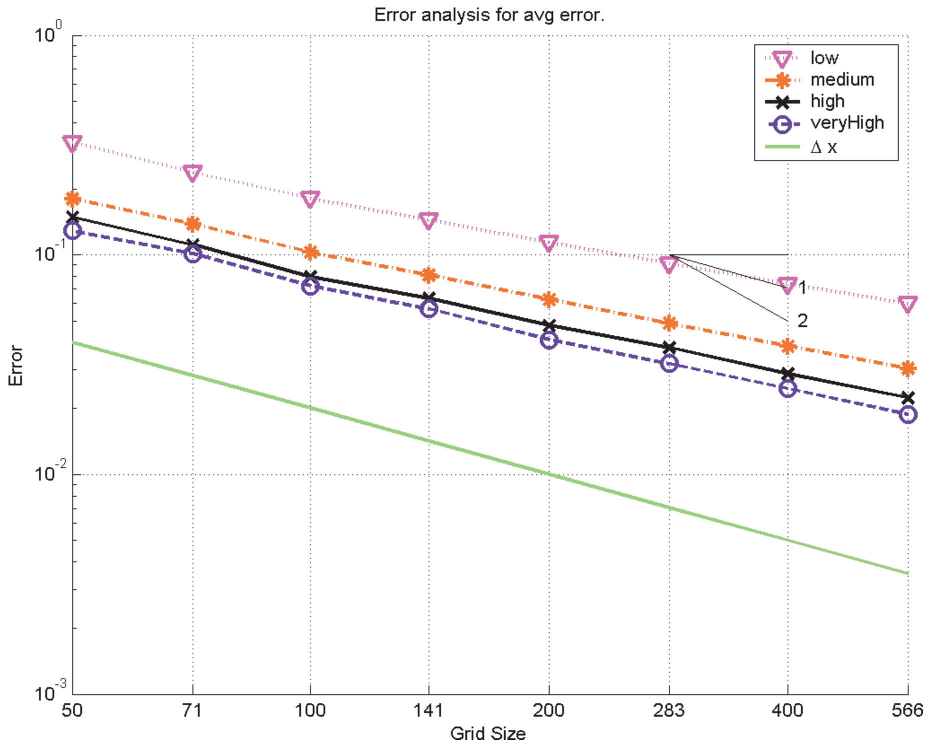
Truth in Advertising

- Comparison to analytic solution not very good
  - But difficult to compare quantitatively to other algorithms



compare approx (solid) to analytic (dashed)



average error for various schemes vs grid size (green line)

Additional Slides?

- HSCC 04 air3D example with weird control policy
- Future work

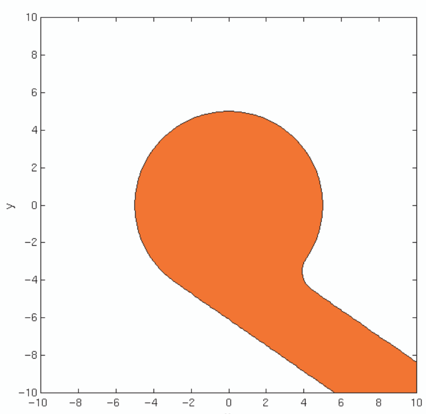
Implicit Surface Functions

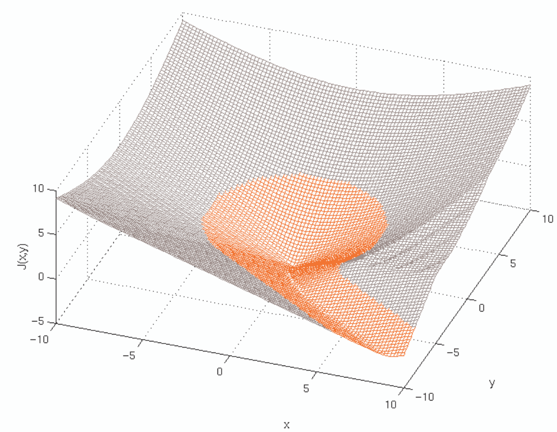
- Surface $\mathcal{S}(t)$ and/or set $G(t)$ are defined implicitly by an isosurface of a scalar function $\phi(x,t)$, with several benefits
  - State space dimension does not matter conceptually
  - Surfaces automatically merge and/or separate
  - Geometric quantities are easy to calculate

$$\phi : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$$

$$\mathcal{G}(t) = \{x \in \mathbb{R}^n \mid \phi(x,t) \leq 0\}$$

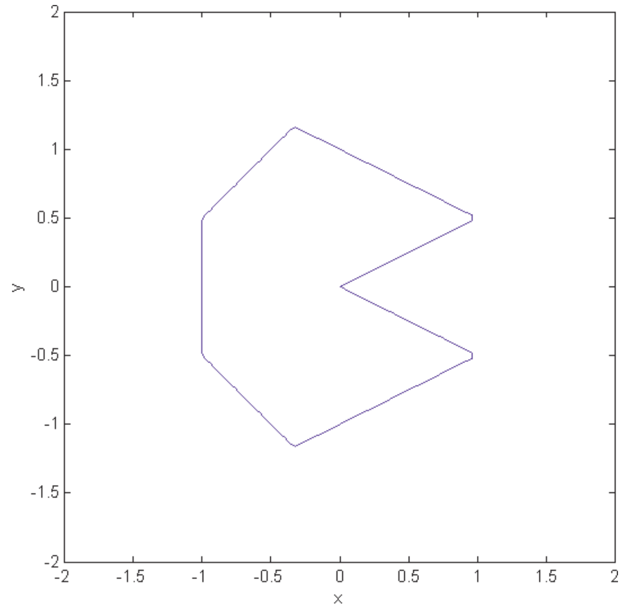
$$\mathcal{S}(t) = \partial\mathcal{G}(t) = \{x \in \mathbb{R}^n \mid \phi(x,t) = 0\}$$

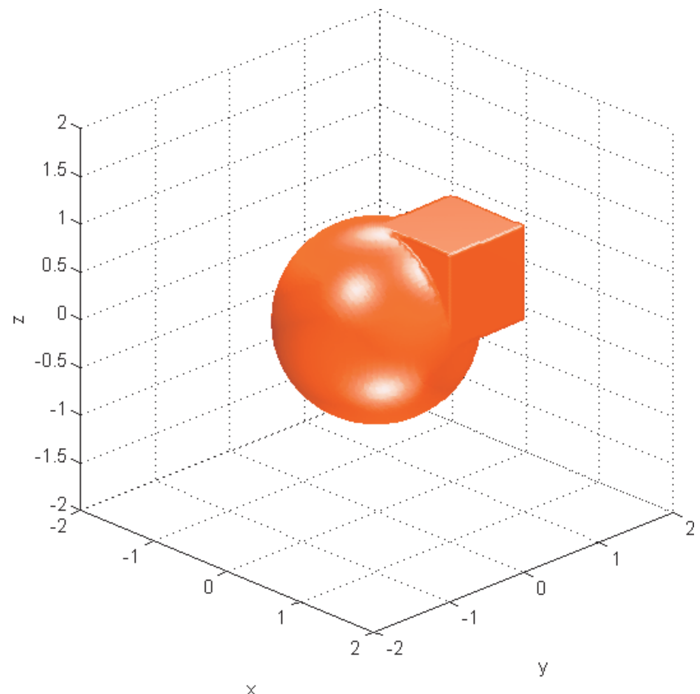




Constructive Solid Geometry

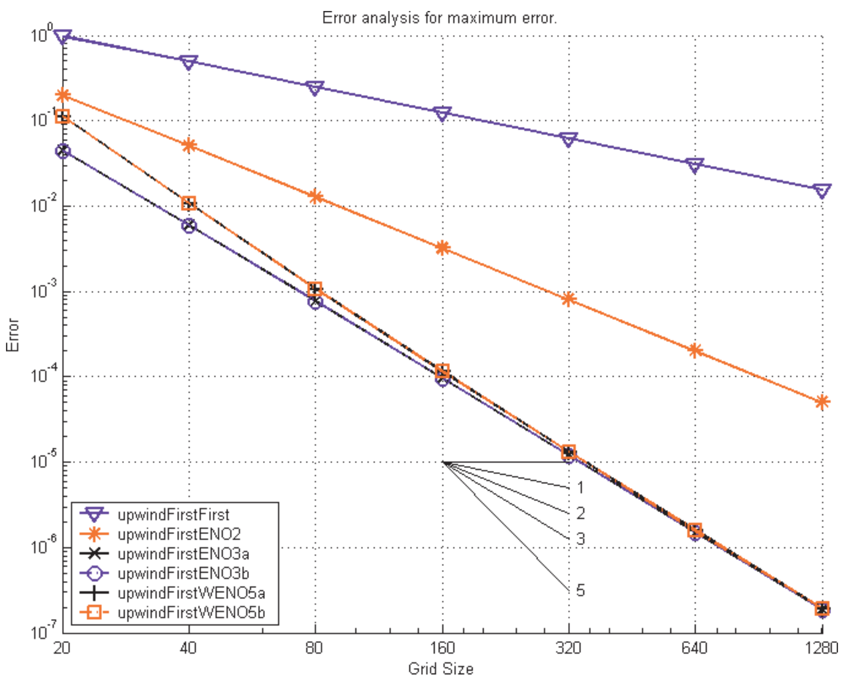
- Simple geometric shapes have simple algebraic implicit surface functions
  - Circles, spheres, cylinders, hyperplanes, rectangles
- Simple set operations correspond to simple mathematical operations on implicit surface functions
  - Intersection, union, complement, set difference



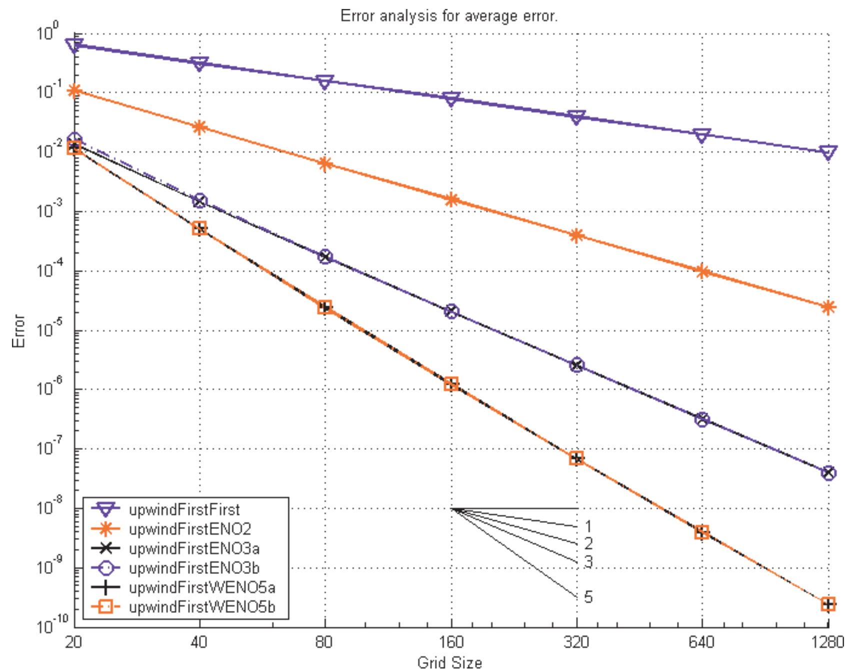


High Order Accuracy

- Temporally: explicit, Total Variation Diminishing Runge-Kutta integrators of order one to three
- Spatially: (Weighted) Essentially Non-Oscillatory upwind finite difference schemes of order one to five
  - Example: approximate derivative of function with kinks



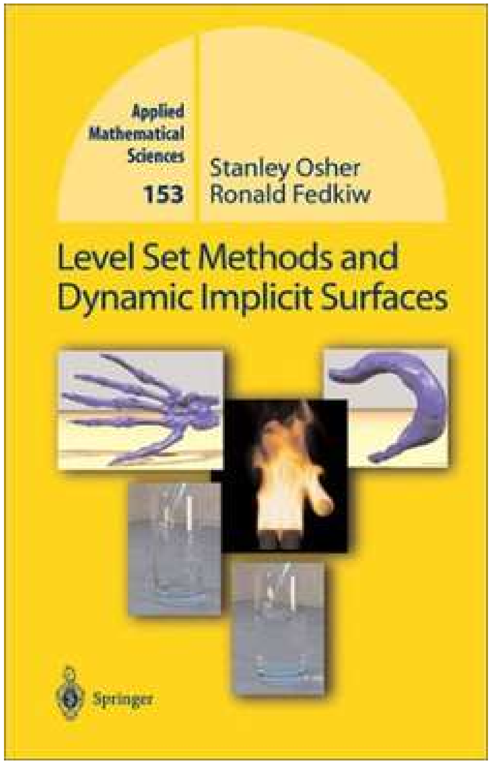
maximum error

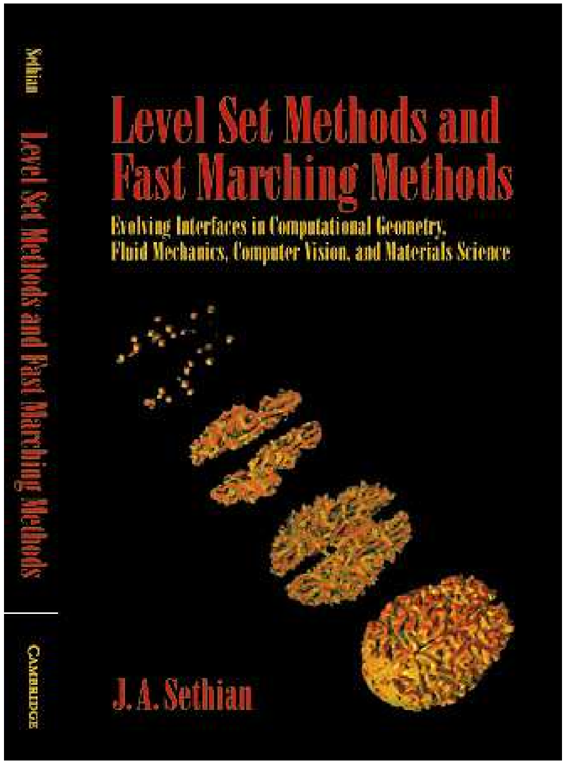


average error

The Toolbox is not a Tutorial

- Users will need to reference the literature
- Two textbooks are available
  - Osher & Fedkiw (2002)
  - Sethian (1999)





Why Use It?

- Dynamic implicit surfaces and Hamilton-Jacobi equations have many practical applications
- The toolbox provides an environment for exploring and experimenting with level set methods
  - Fourteen examples
  - Approximations of most common types of motion
  - High order accuracy
  - Arbitrary dimension
  - Reasonable speed with vectorized code
  - Direct access to Matlab debugging and visualization
  - Source code for all toolbox routines
- The toolbox is free

`http://www.cs.ubc.ca/~mitchell/ToolboxLS`

Under development

- PDE terms
  - More general Dirichlet and Neumann boundary conditions
  - Fast signed distance function construction
- Other methods
  - Implicit temporal integrators
  - Static Hamilton-Jacobi
  - Vector level set methods
  - Particle level set methods
- More application examples
  - Hybrid system reachable sets
  - Image processing