Homework #4

Keep your paper and pencil answers brief. For simulations, submit a **labeled** plot and your **documented** MATLAB files. In addition to the standard labels and documentation, each plot and MATLAB file should explicitly state your name and to which question they apply.

1. **Stable + Stable = Unstable** (this question was moved from Homework #3): Consider the following hybrid system $\mathcal{H}_2$. Let

$$D_+ = \left\{ \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in \mathbb{R}^2 \,\middle|\, x_1 x_2 \geq 0 \right\},$$
$$D_- = \left\{ \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in \mathbb{R}^2 \,\middle|\, x_1 x_2 \leq 0 \right\}.$$

On D_s (for $s \in \{+, -\}$) the dynamics are governed by the differential equation $\dot{x} = F_s(x)$, where $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ and the vector fields F_+ and F_- are defined as follows:

$$F_+(x) = \begin{bmatrix} -x_1 + 100x_2 \\ -10x_1 - x_2 \end{bmatrix}, \qquad F_-(x) = \begin{bmatrix} -x_1 + 10x_2 \\ -100x_1 - x_2 \end{bmatrix}.$$

When a trajectory of F_+ hits the boundary of D_+, it starts following F_- until the boundary of D_- is hit at which point it switches back to F_+, etc.

   (a) Sketch a graphical representation of this hybrid system, showing its modes, vector fields, domains, edges and guards.

   (b) Using paper and pencil, show that the differential equation $\dot{x} = F_s(x)$ (for each $s \in \{+, -\}$), when considered on the whole plane, has a unique equilibrium. Using Lyapunov analysis, show that this equilibrium is stable (hint: for $F_+(x)$, try $V_+(x) = x^T P_+ x$ where $P_+ = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 10 \end{smallmatrix}\right]$).

   (c) Using a simulation, show that $\mathcal{H}_2$ is unstable. **Bonus:** Demonstrate the same thing analytically (hint: a system that is unstable in forward time is stable in backward time).

   (d) By combining two stable systems, we have generated an unstable hybrid system. Can you give a similar example of unstable + unstable = stable? What about stable + unstable = stable?

2. **Temporal Logics.** Translate each of the following formulas into a form which uses only $\wedge$, $\vee$, $\neg$, **EX**, **EG** and/or **EU**.

   (a) $\mathbf{AF}(p \vee \mathbf{AG}\neg q)$.

   (b) $\mathbf{A}[(x = y)\ \mathbf{R}\ z]$.

   (c) $\mathbf{EF}(p \implies q)$.

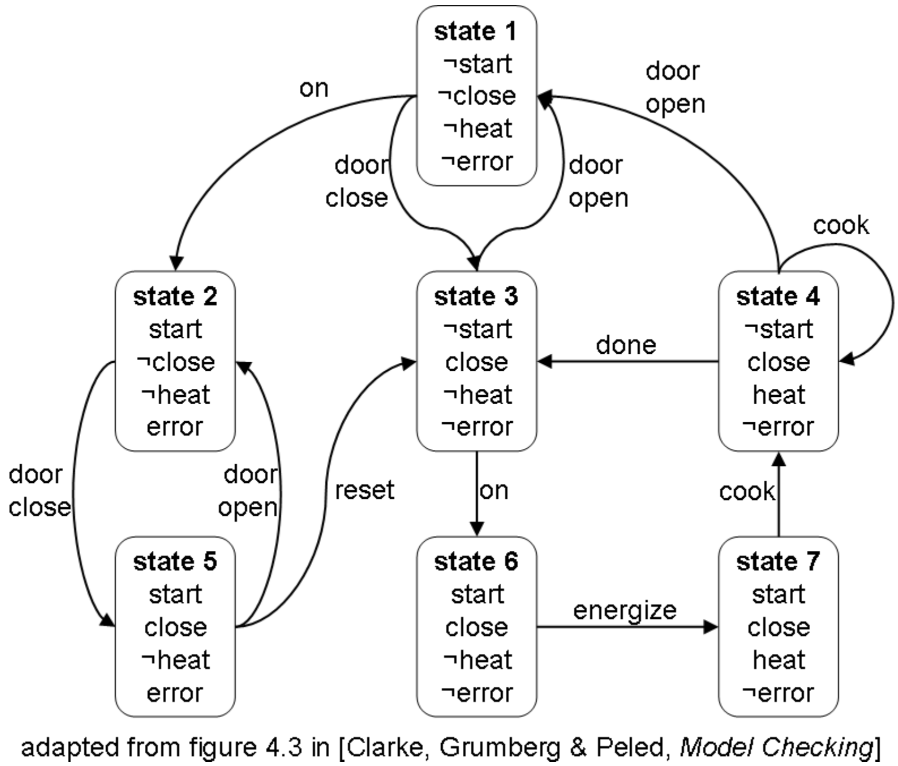

Figure 1: Microwave model.

3. **Model Checking.** For the microwave model given in figure 1, determine by model checking whether the CTL formula **EF**(error $\wedge$ **EG** close) holds in state 1. If you wish to avoid excessive writing, you may use "s", "c", "h", "e" to refer to propositions "start", "close", "heat" and "error" respectively, and numbers instead of the state labels (eg: "5" instead of "state 5").

4. **Write a CTL Model Checking Algorithm.** Assume that for model $M = (S, S_0, R)$ we have labelled the states in S where CTL formula f holds.

   (a) Write a model checking algorithm for the CTL formula $\mathbf{EF}f$.

   (b) What is the $\mathcal{O}()$ ("big-O") complexity of your algorithm in terms of $|S|$, $|S_0|$ and/or $|R|$ (the sizes of the sets)?