# Analyzing Games: Nash Equilibria

ISCI 330 Lecture 5

January 23, 2006

# Lecture Overview

1. Recap
2. Best Response and Nash Equilibrium
3. Mixed Strategies

# Pareto Optimality

- Idea: sometimes, one outcome $o$ is at least as good for every agent as another outcome $o'$, and there some agent who strictly prefers $o$ to $o'$
  - in this case, it seems reasonable to say that $o$ is better than $o'$
  - we say that $o$ Pareto-dominates $o'$.
- An outcome $o^*$ is Pareto-optimal if there is no other outcome that Pareto-dominates it.

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- Let $a_{-i} = \langle a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \rangle$.
  - now $a = (a_{-i}, a_i)$
- Best response: $a_i^* \in BR(a_{-i})$ iff $\forall a_i \in A_i,\ u_i(a_i^*, a_{-i}) \geq u_i(a_i, a_{-i})$

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- What can we say about which actions will occur?

- Idea: look for stable action profiles.
- $a = \langle a_1, \ldots, a_n \rangle$ is a Nash equilibrium iff $\forall i,\ a_i \in BR(a_{-i})$.

# Nash Equilibria of Example Games

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|---|---|---|
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*The paradox of Prisoner's dilemma: the Nash equilibrium is the only non-Pareto-optimal outcome!*

# Lecture Overview

# Mixed Strategies

- It would be a pretty bad idea to play any deterministic strategy in matching pennies
- Idea: confuse the opponent by playing randomly
- Define a strategy $s_i$ for agent $i$ as any probability distribution over the actions $A_i$.
    - pure strategy: only one action is played with positive probability
    - mixed strategy: more than one action is played with positive probability
        - these actions are called the support of the mixed strategy
- Let the set of all strategies for $i$ be $S_i$
- Let the set of all strategy profiles be $S = S_1 \times \ldots \times S_n$.

# Utility under Mixed Strategies

- What is your payoff if all the players follow mixed strategy profile $s \in S$?
  - We can't just read this number from the game matrix anymore: we won't always end up in the same cell

# Utility under Mixed Strategies

- What is your payoff if all the players follow mixed strategy profile $s \in S$?
  - We can't just read this number from the game matrix anymore: we won't always end up in the same cell
- Instead, use the idea of expected utility from decision theory:

$$u_i(s) = \sum_{a \in A} u_i(a) Pr(a|s)$$

$$Pr(a|s) = \prod_{j \in N} s_j(a_j)$$

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Our definitions of best response and Nash equilibrium generalize from actions to strategies.

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- Nash equilibrium:
  - $s = \langle s_1, \ldots, s_n \rangle$ is a Nash equilibrium iff $\forall i$, $s_i \in BR(s_{-i})$
- Every finite game has a Nash equilibrium! [Nash, 1950]
  - e.g., matching pennies: both players play heads/tails 50%/50%