Complexity of Nash Equilibrium

David R.M. Thompson
(+ slides by Constantinos Daskalakis)

January 17, 2013

Outline

Complexity Recap

Definition (P)

The set of decision problems that can be solved in polynomial time by a deterministic Turing machine.
e.g., is this list sorted?

Definition (NP)

The set of decision problems that can be solved in polynomial time by a non-deterministic Turing machine.
e.g., is this boolean formula satisfiable?

Complexity Recap

Definition (Reduction)

Transforming one problem into another (using a deterministic Turing machine).
$A \leq_P B$ means "Problem A can be solved using an algorithm for problem B, with polynomial additional cost."

- $A \leq_P B$ and $B \in NP$ implies $A \in NP$.

Complexity Recap

Definition (X-hard)

A problem is X-hard iff it is at least as hard as any problem in X.

- $A \leq_P B$ and A is NP-hard implies B is NP-hard.

Definition (X-complete)

A problem is X-complete iff it is in X and X-hard.

- $A \leq_P B$, $B \leq_P A$ and A is NP-complete implies B is NP-complete.

Where does Nash fit in?

- As a decision problem, it's easy:
  Does this game have a Nash equilibrium? Yes!

Where does Nash fit in?

- As a decision problem, it's easy:
  Does this game have a Nash equilibrium? Yes!
- Ask slightly more and it becomes NP-complete, e.g.,
  - Does this game have more than one Nash equilibrium?
  - Does this game have a Nash equilibrium equilibrium where action a_i is played with non-zero probability?
  - Does this game have a Nash equilibrium equilibrium where action a_i is played with zero probability?
- But what's the complexity of finding a Nash equilibrium?

Outline

Where does Nash fit in?

- What's the complexity of finding a Nash equilibrium?

Definition (FNP)

The set of function problems that can be solved in polynomial time by a non-deterministic Turing machine.
e.g., find a satisfying assignment for this boolean formula.

- ϵ-NASH $\in$ FNP.

Where does Nash fit in?

- What's the complexity of finding a Nash equilibrium?

Definition (FNP)

The set of function problems that can be solved in polynomial time by a non-deterministic Turing machine.
e.g., find a satisfying assignment for this boolean formula.

- ϵ-NASH $\in$ FNP.
- What's that ϵ mean?

Where does Nash fit in?

- What's the complexity of finding a Nash equilibrium?

Definition (FNP)

The set of function problems that can be solved in polynomial time by a non-deterministic Turing machine.
e.g., find a satisfying assignment for this boolean formula.

- ϵ-NASH $\in$ FNP.
- What's that ϵ mean?
- Where did the ϵ come from? Games with more than two players might not any rational-valued Nash equilibrium.

Where does Nash fit in?

Definition (PPAD)

The set of function problems where a solution is guaranteed to exist, by a parity argument on a directed graph.

- PPAD $\subseteq$ FNP.

Theorem (Daskalakis et al, Chen & Deng)

ϵ-Nash is PPAD-complete.

Where does Nash fit in?

Definition (PPAD)

The set of function problems where a solution is guaranteed to exist, by a parity argument on a directed graph.

- PPAD $\subseteq$ FNP.

Theorem (Daskalakis et al, Chen & Deng)

ϵ-Nash is PPAD-complete.

- Agenda:
  - Show ϵ-NASH $\leq_P$ BROUWER (PPAD-complete)
    i.e., ϵ-NASH $\in$ PPAD
  - Show BROUWER $\leq_P$ ϵ-NASH
    i.e., ϵ-NASH is PPAD-hard.

Outline

Nash's Theorem "$\Rightarrow$" NASH $\in$ PPAD

Nash $\longrightarrow$ Brouwer

Kick / Dive	Left	Right
Left	1 , -1	-1 , 1
Right	-1 , 1	1, -1

Penalty Shot Game

$\longmapsto$

$f: [0,1]^2 \to [0,1]^2$, cont.
such that
fixed point $\equiv$ Nash eq.

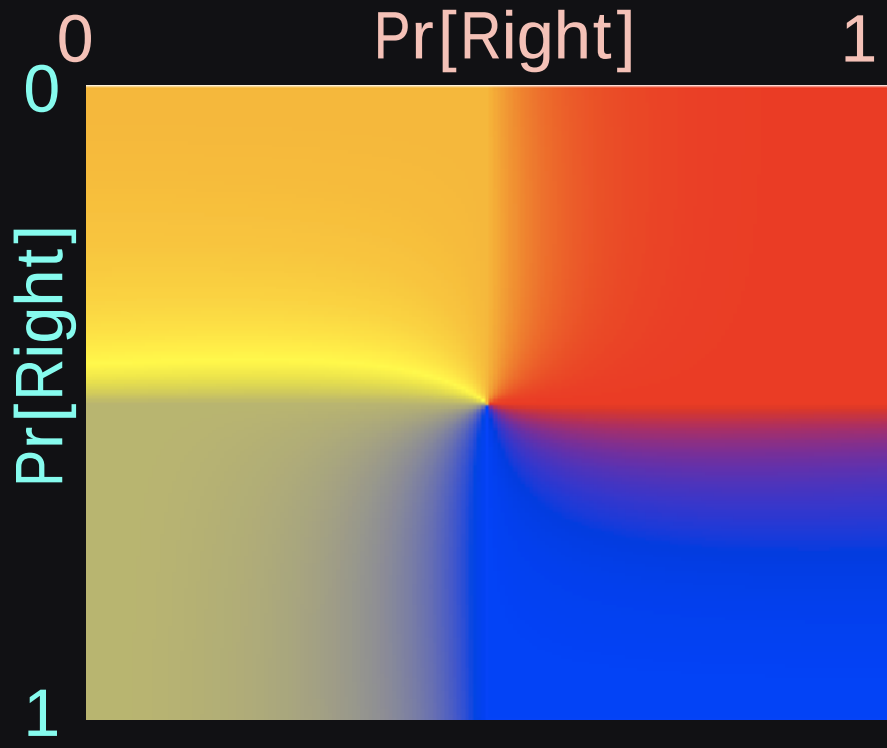
Nash's Theorem "⇒" NASH ∈ PPAD

Nash ⟶ Brouwer

Kick / Dive	Left	Right
Left	1 , -1	-1 , 1
Right	-1 , 1	1, -1

Penalty Shot Game

⟼

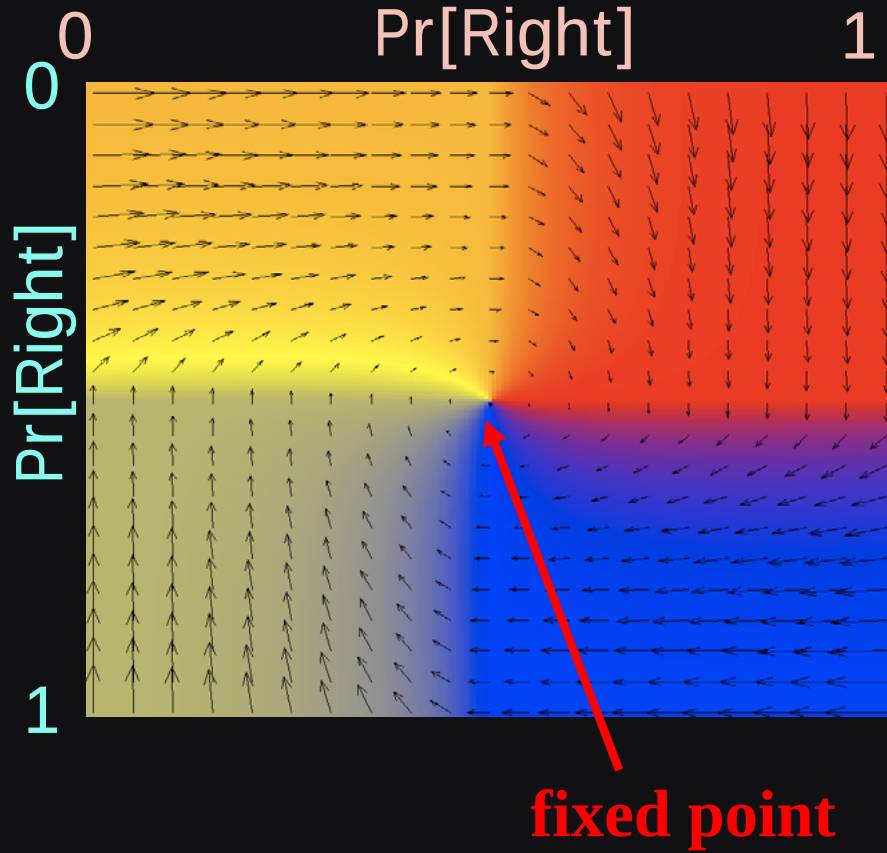


Nash's Theorem "⇒" NASH ∈ PPAD

Nash ⟶ Brouwer

		½ Left	½ Right
	Kick / Dive		
½	Left	1 , -1	-1 , 1
½	Right	-1 , 1	1, -1

Penalty Shot Game



Nash's Theorem "⇒" NASH ∈ PPAD

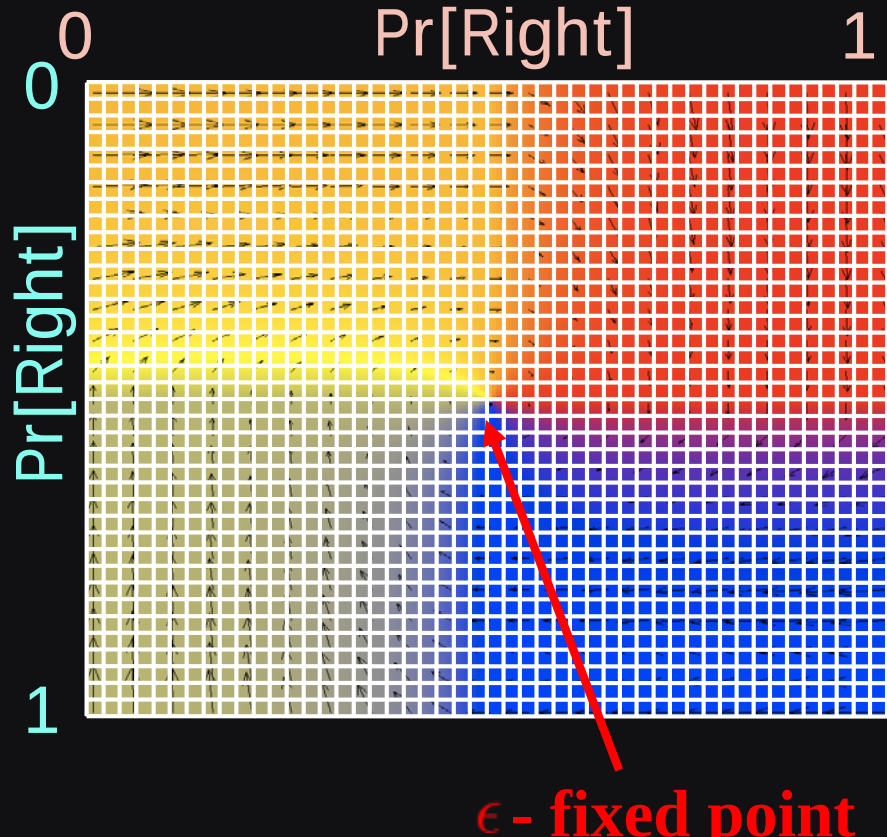
REAL PROOF

Nash ⟶ Brouwer

		½	½
	Kick / Dive	Left	Right
½	Left	1 , -1	-1 , 1
½	Right	-1 , 1	1, -1

Penalty Shot Game

⟼



Outline

PPAD-Hardness of NASH [DGP ’05]

Nash ⟵ Brouwer

game whose Nash equilibria are close to the fixed points of f ⟵| $f: [0,1]^3 \to [0,1]^3$, continuous & p.w.linear

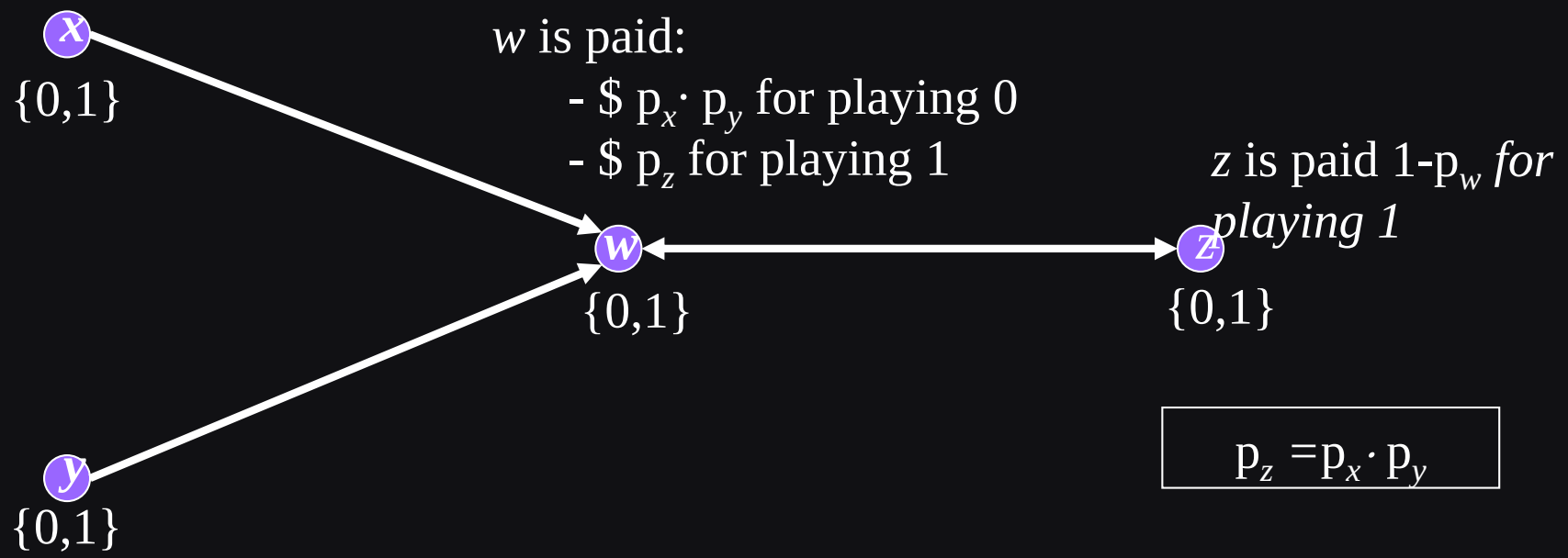
- *Game-gadgets*: games acting as arithmetic gates

Games that do real arithmetic

e.g. multiplication game *(similarly addition, subtraction)*

two strategies per player, say {0,1};

➡ Mixed strategy ≡ a number in [0,1]
(probability of playing 1)



Games that do real arithmetic

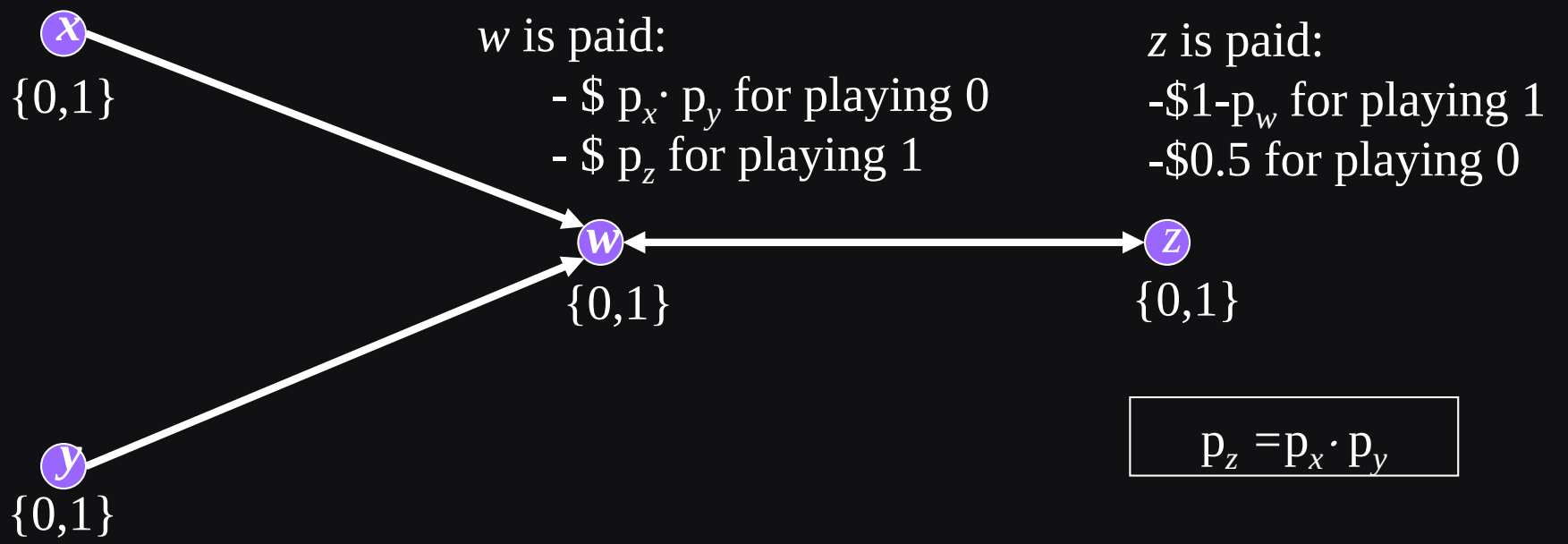
w's payoff

for playing 0

	y plays 0	y plays 1
x plays 0	0	0
x plays 1	0	1

for playing 1

z plays 0	0
z plays 1	1



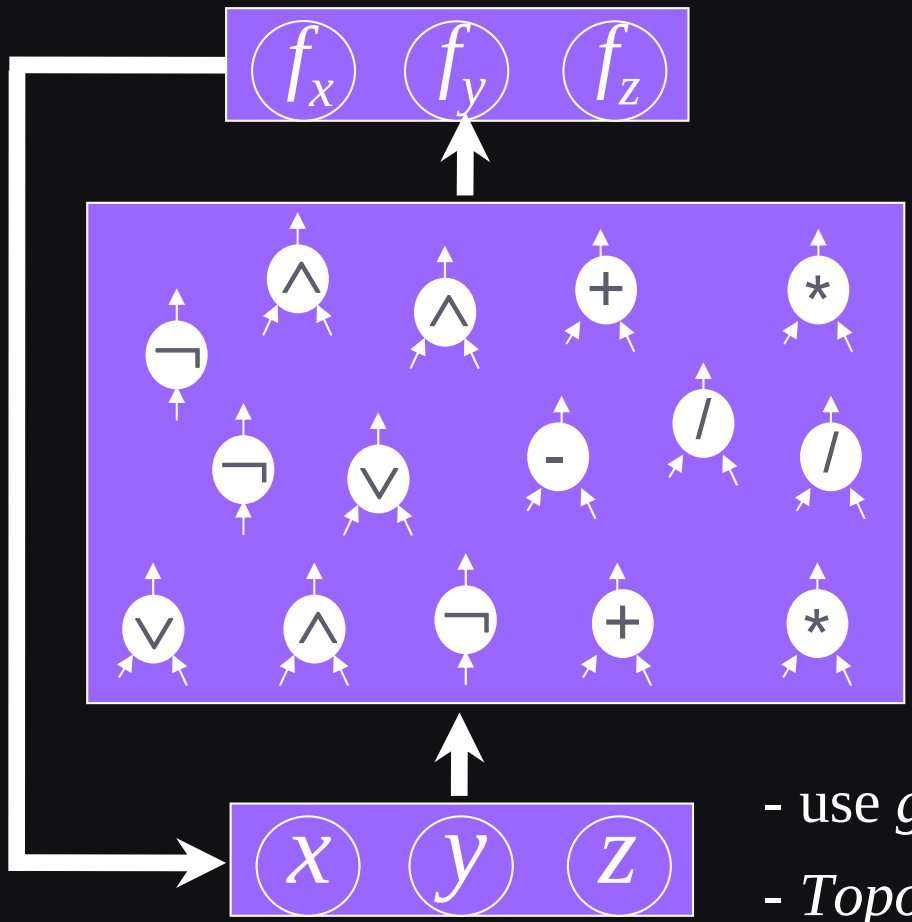
w is paid:
- $ $p_x \cdot p_y$ for playing 0
- $ p_z for playing 1

z is paid:
-\$1-p_w for playing 1
-\$0.5 for playing 0

$$p_z = p_x \cdot p_y$$

PPAD-Hardness of NASH [DGP ’05]

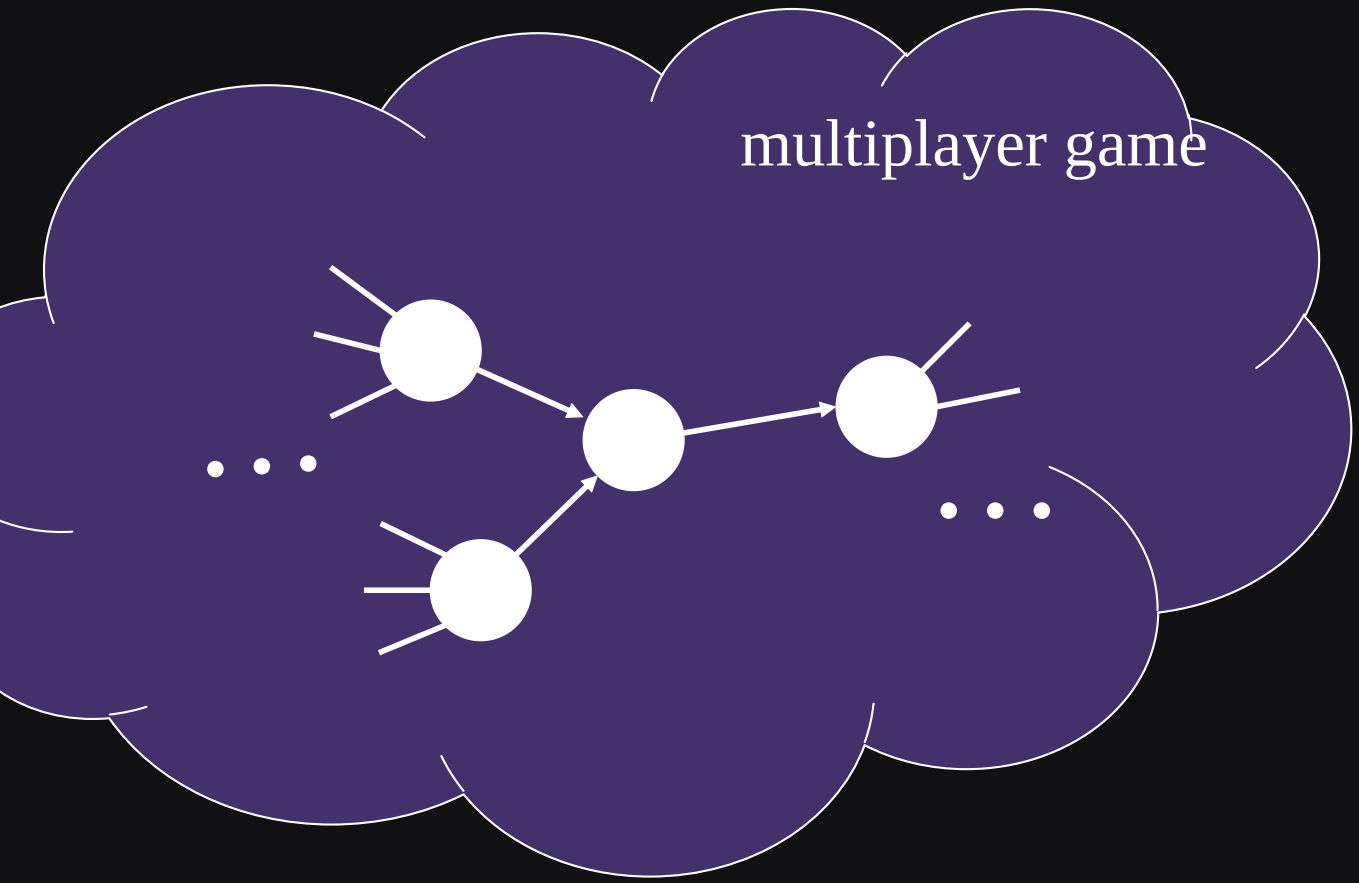
Nash ⟵ Brouwer



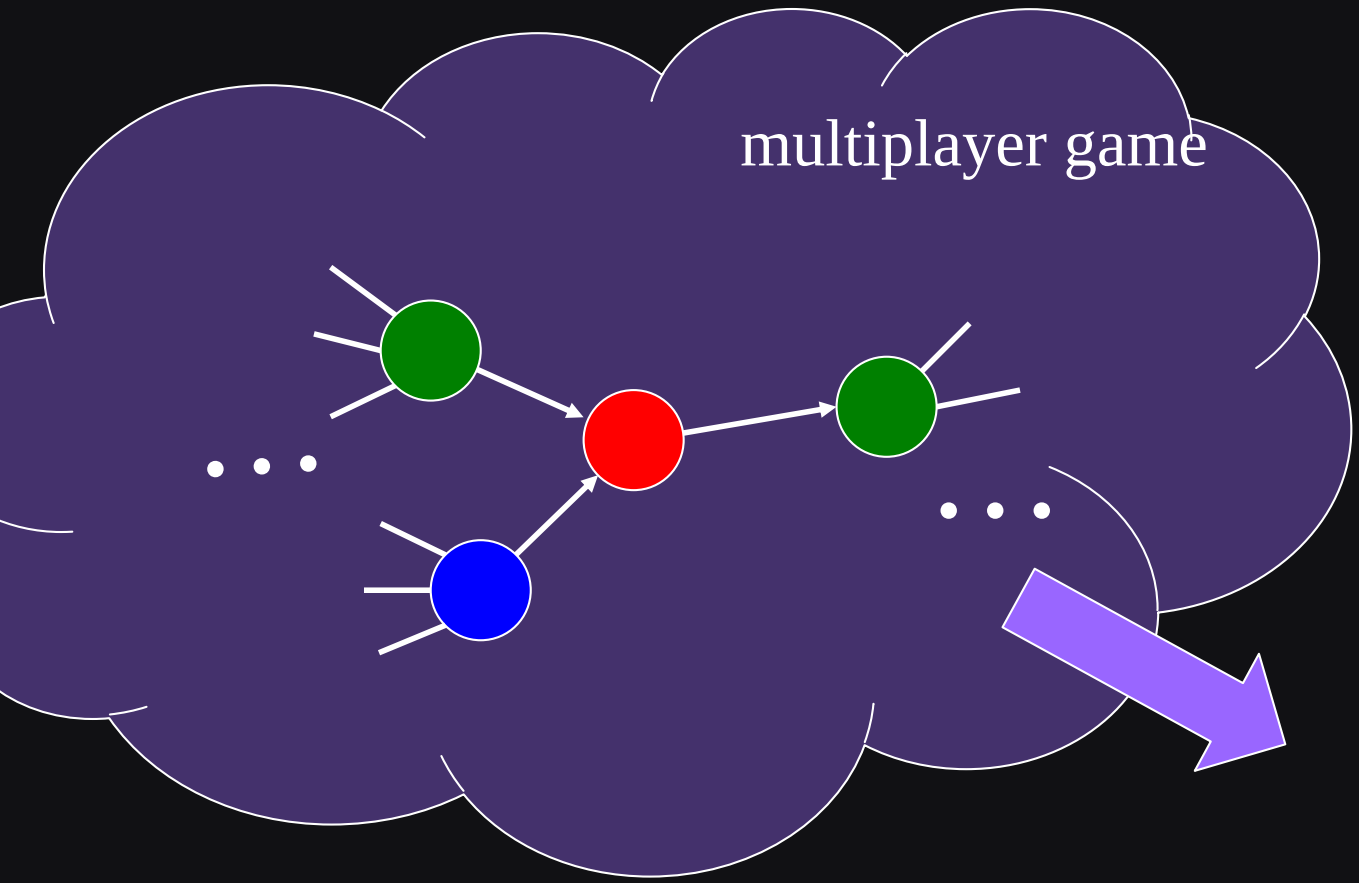
f: $[0,1]^3 \to [0,1]^3$,
continuous & p.w.linear

- use *game-gadgets* to simulate f with a game
- *Topology*: noise reduction

Reduction to 3 players [Das, Pap ‘05]

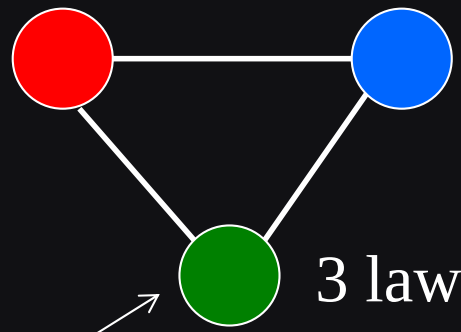


Reduction to 3 players [Das, Pap ‘05]



“represents” **red** players

“represents” **blue** players

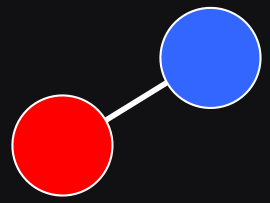


3 lawyers

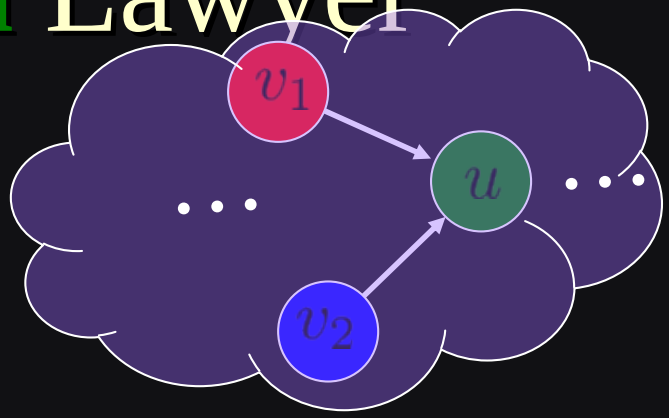
Coloring: no two nodes affecting one another, or affecting the same third player use the same color;

“represents” all **green** players

Payoffs of the **Green** Lawyer



	$v_2 : 0$	$v_2 : 1$	$\neq v_2$
$v_1 : 0$	copy of the payoff table of node *u*		0
$v_1 : 1$			0
$\neq v_1$	0	0	0

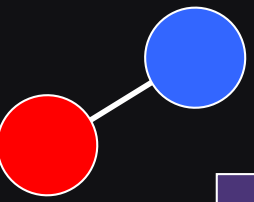


payoffs of the **green** lawyer for representing node *u*

wishful thinking: The Nash equilibrium of the lawyer-game, gives a Nash equilibrium of the original multiplayer game, after marginalizing with respect to individual nodes.

But why would a lawyer represent every node equally?

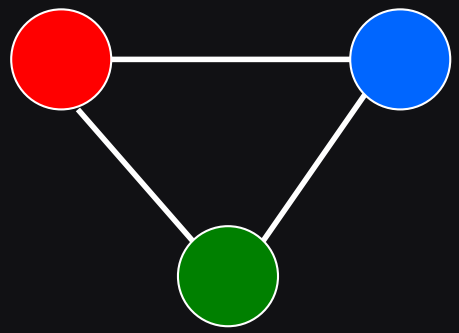
Enforcing Fairness



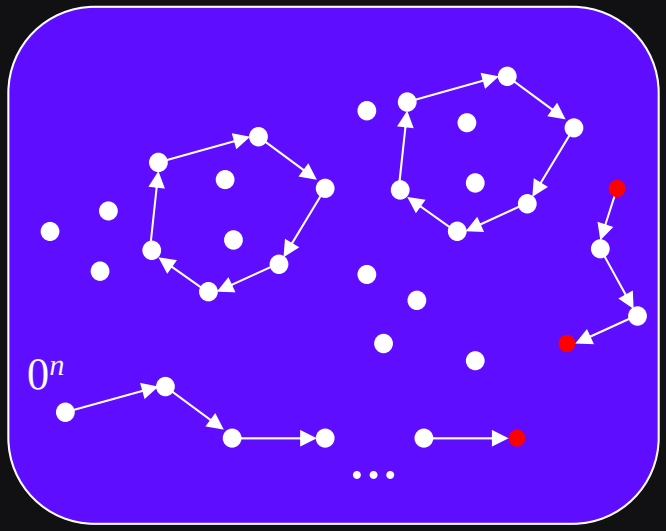
	$v_2 : 0$	$v_2 : 1$	$\neq v_2$
$v_1 : 0$	copy of the payoff table of node *u*		0
$v_1 : 1$			0
$\neq v_1$	0	0	0

+

lawyers play on the side a high-stakes game over the nodes they represent



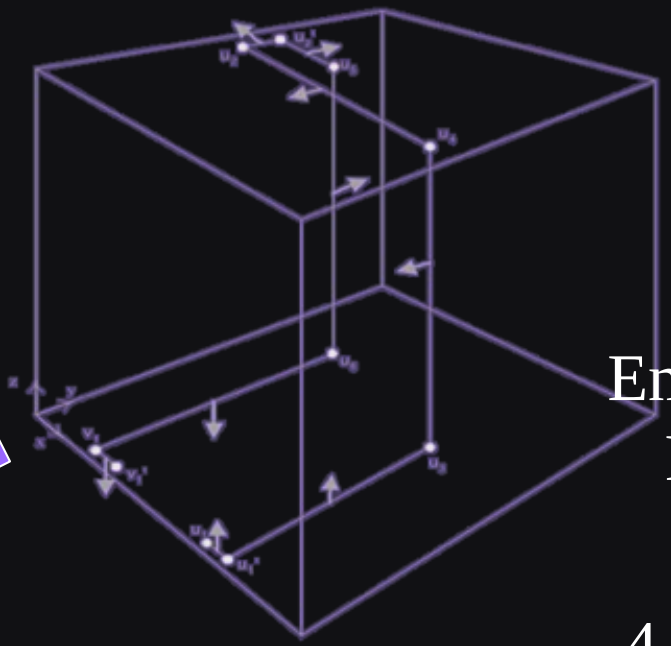
PPAD-hardness of NASH



Generic PPAD

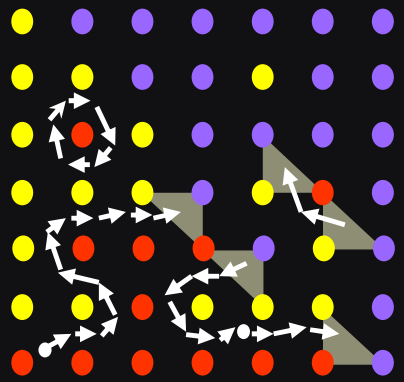
[Pap ’94]

[DGP ’05]



Embedded PPAD

[DGP ’05]



SPERNER

[DGP ’05]

p.w. linear BROUWER

[DGP ’05]

multi-player NASH

[DGP ’05]

4-player NASH

[DP ’05]

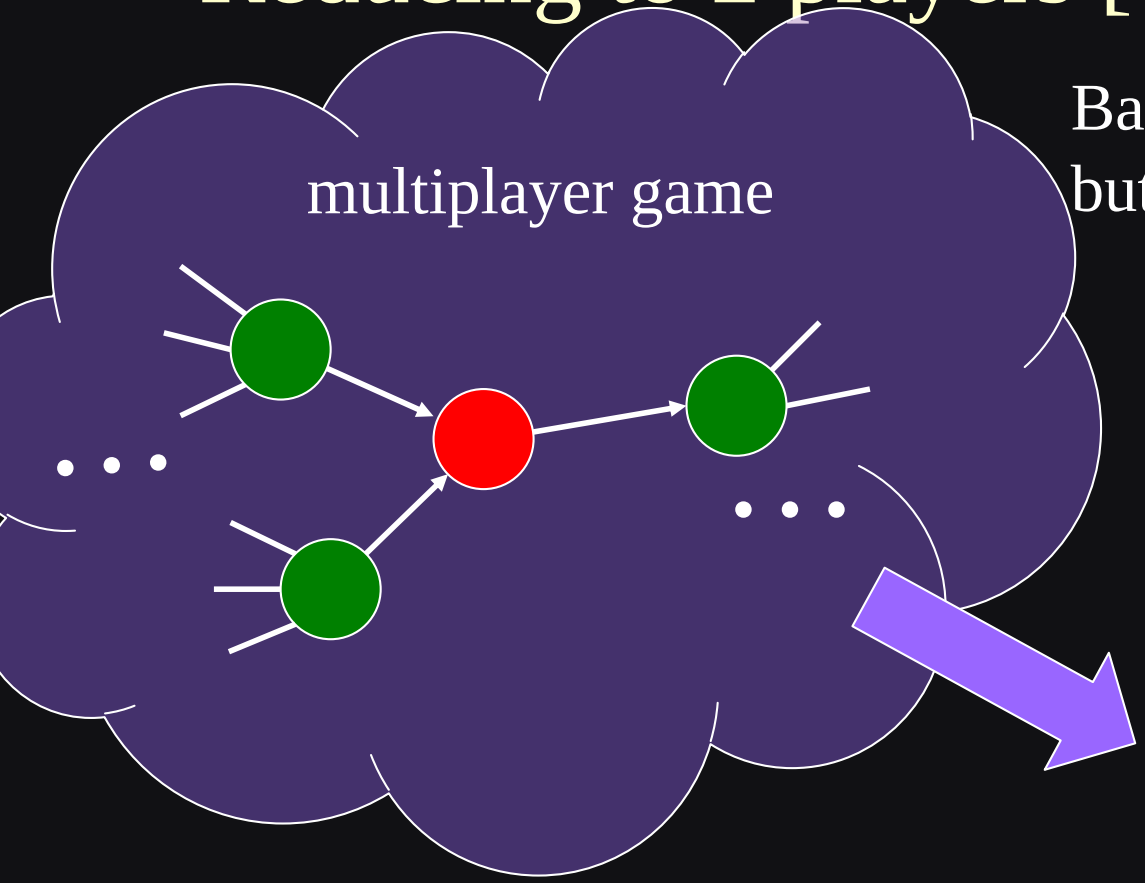
[CD’05]

3-player NASH

[CD’05]

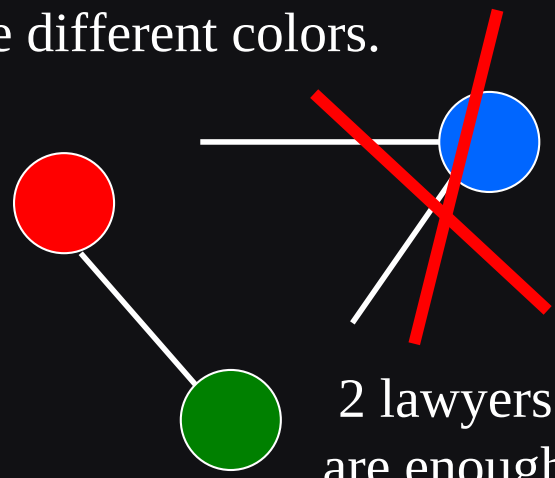
2-player NASH

Reducing to 2 players [Chen, Deng ’05]



Based on the following simple, but crucial observation:

- the expected payoff of each lawyer is additive w.r.t. the nodes that another lawyer represents;
- hence, if two nodes affect the same third node, they don’t need to have different colors.



2 lawyers are enough

Coloring: no two nodes affecting one another, ~~or affecting the same third player~~ use the same color;

two colors suffice to color the multiplayer game in the [DGP 05] construction