Mixed Strategies and Nash Equilibrium

Game Theory Course:
Jackson, Leyton-Brown & Shoham

Mixed Strategies

- It would be a pretty bad idea to play any deterministic strategy in matching pennies
- Idea: confuse the opponent by playing randomly
- Define a strategy s_i for agent i as any probability distribution over the actions A_i.
  - pure strategy: only one action is played with positive probability
  - mixed strategy: more than one action is played with positive probability
    - these actions are called the support of the mixed strategy
- Let the set of all strategies for i be S_i
- Let the set of all strategy profiles be $S = S_1 \times \ldots \times S_n$.

Utility under Mixed Strategies

- What is your payoff if all the players follow mixed strategy profile $s \in S$?
  - We can't just read this number from the game matrix anymore: we won't always end up in the same cell

Utility under Mixed Strategies

- What is your payoff if all the players follow mixed strategy profile $s \in S$?
  - We can't just read this number from the game matrix anymore: we won't always end up in the same cell
- Instead, use the idea of expected utility from decision theory:

$$u_i(s) = \sum_{a \in A} u_i(a) Pr(a|s)$$

$$Pr(a|s) = \prod_{j \in N} s_j(a_j)$$

Best Response and Nash Equilibrium

Our definitions of best response and Nash equilibrium generalize from actions to strategies.

Definition (Best response)

$s_i^* \in BR(s_{-i})$ iff $\forall s_i \in S_i,\ u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$

Definition (Nash equilibrium)

$s = \langle s_1, \ldots, s_n \rangle$ is a Nash equilibrium iff $\forall i,\ s_i \in BR(s_{-i})$

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Theorem (Nash, 1950)

Every finite game has a Nash equilibrium.

Computing Mixed Nash Equilibria

Battle of the Sexes

	B	F
B	2, 1	0, 0
F	0, 0	1, 2

- It's hard in general to compute Nash equilibria, but it's easy when you can guess the support
- For BoS, let's look for an equilibrium where all actions are part of the support

Computing Mixed Nash Equilibria

Battle of the Sexes

	B	F
B	2, 1	0, 0
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- Let player 2 play B with p, F with $1 - p$.
- If player 1 best-responds with a mixed strategy, player 2 must make him indifferent between F and B (why?)

Computing Mixed Nash Equilibria

Battle of the Sexes

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- Let player 2 play B with p, F with $1 - p$.
- If player 1 best-responds with a mixed strategy, player 2 must make him indifferent between F and B (why?)

$$u_1(B) = u_1(F)$$

$$2p + 0(1 - p) = 0p + 1(1 - p)$$

$$p = \frac{1}{3}$$

Computing Mixed Nash Equilibria

Battle of the Sexes

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- Likewise, player 1 must randomize to make player 2 indifferent.
  - Why is player 1 willing to randomize?

Computing Mixed Nash Equilibria

Battle of the Sexes

	B	F
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- Likewise, player 1 must randomize to make player 2 indifferent.
  - Why is player 1 willing to randomize?
- Let player 1 play B with q, F with $1 - q$.

$$u_2(B) = u_2(F)$$

$$q + 0(1-q) = 0q + 2(1-q)$$

$$q = \frac{2}{3}$$

- Thus the mixed strategies $(\frac{2}{3}, \frac{1}{3})$, $(\frac{1}{3}, \frac{2}{3})$ are a Nash equilibrium.

Interpreting Mixed Strategy Equilibria

What does it mean to play a mixed strategy? Different interpretations:

- Randomize to confuse your opponent
  - consider the matching pennies example
- Randomize when uncertain about the other's action
  - consider battle of the sexes
- Mixed strategies are a concise description of what might happen in repeated play: count of pure strategies in the limit
- Mixed strategies describe population dynamics: 2 agents chosen from a population, all having deterministic strategies. MS gives the probability of getting each PS.

Example - Soccer Penalty Kicks

$Kicker/Goalie$	$Left$	$Right$
$Left$	$0, 1$	$1, 0$
$Right$	$1, 0$	$0, 1$

Example - Soccer Penalty Kicks

Kicker/Goalie	*Left*	*Right*
Left	0, 1	1, 0
Right	.75, .25	0, 1

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Hardness beyond 2×2 games

Algorithms

Two example algorithms for finding NE

- LCP (Linear Complementarity) formulation
  - [Lemke-Howson '64]
- Support Enumeration Method
  - [Porter et al. '04]

The Lemke-Howson Algorithm

CPSC 532L

Lecture Overview

Linear Programming

A linear program is defined by:

- a set of real-valued variables
- a linear objective function
  - a weighted sum of the variables
- a set of linear constraints
  - the requirement that a weighted sum of the variables must be greater than or equal to some constant

Linear Programming

Given n variables and m constraints, variables x and constants w, a and b:

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{n} w_i x_i \\ \text{subject to} \quad & \sum_{i=1}^{n} a_{ij} x_i \leq b_j \qquad \forall j = 1 \ldots m \end{aligned}$$

- These problems can be solved in polynomial time using interior point methods.
  - Interestingly, the (worst-case exponential) simplex method is often faster in practice.

Lecture Overview

1. Linear Programming
2. Lemke-Howson Algorithm

Two-player equilibrium constraints

$$\sum_{a_2 \in A_2} u_1(a_1, a_2) \cdot s_2(a_2) + r_1(a_1) = U_1^* \qquad \forall a_1 \in A_1$$

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$$\sum_{a_i \in A_i} s_i(a_i) = 1 \qquad \forall i \in N$$

$$s_i(a_i) \geq 0 \qquad \forall i \in N, a_i \in A_i$$

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- We can write down a set of constraints that a two player strategy profile satisfies if and only if it is a Nash equilibrium.

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- U_i^* is the utility of i's best responses.
- $s_i(a_i)$ is the probability that i plays a_i.
- $r_i(a_i)$ is a "slack" variable.
- Each $u_i(a_i, a_{-i})$ is a constant.

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- s_1 and s_2 are valid probability distributions.

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- Slack variables $r_i(a_i)$ are non-negative.
- U_1^* is weakly greater than the EU of any of player 1's actions, given $s_2 \ldots$
- and exactly equal to the EU of every action in the support.

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- So we're done! Or are we?
- This requirement changes the problem from a linear program to a linear complementarity program.
- Unfortunately, there is no general algorithm for solving LCPs.

Mixed strategy labels

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It uses a concept of labels on mixed strategies.

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Definition (Labels)

Every possible mixed strategy s_i is given a set of labels $L(s_i) \subseteq A_1 \cup A_2$. The strategy s_i has the following labels:

- Every action $a_i \in A_i$ satisfying $s_i(a_i) = 0$, and
- Every action $a_{-i} \in A_{-i}$ such that $a_{-i} \in BR_{-i}(s_i)$.

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A pair of strategies (s_1, s_2) is a Nash equilibrium iff it is completely labelled: $L(s_1) \cup L(s_2) = A_1 \cup A_2$.

Searching for a completely labelled pair

- The Lemke-Howson algorithm can be understood as searching the two spaces of labelled strategies for a fully-labelled pair.
- When the game is nondegenerate*, there are no strategies with more labels than an agent has actions.
- So a completely labelled pair of strategies must consist of a pair that has no labels in common.

Pivoting

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- Basic strategy:
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  2. Pivot to a new s_1; its new label must duplicate a label of s_2.
  3. Repeat:
     1. Pivot to a new strategy to remove the duplicated label (the "leaving" label).
     2. If the new label (the "entering" label) is a duplicate, continue.
     3. Otherwise, the "missing" label must have been found. Halt.

Lemke-Howson properties

- Only works on 2-player games. (why?)
- Guaranteed to find at least one equilibrium.
- Not guaranteed to find all equilibria.
- May require exponentially many pivots.
- Quite fast in practice.

The basic idea behind SEM

- If you "guess" the right support, finding an equilibrium only requires solving a system of polynomial inequalities.
- In practice, tools like MINOS [Murtagh, Saunders, 2010] solve these systems quickly.
- To find one (or all) Nash equilibra, just enumerate supports.

Hardness beyond 2×2 games

Support Enumeration Method: Porter *et al.* 2004

- Step 1: Finding a NE with a specific support

$$\sum_{a_{-1} \in \sigma_{-i}} p(a_{-i}) u_i(a_i, a_{-i}) = v_i \qquad \forall i \in \{1,2\}, a_i \in \sigma_i$$

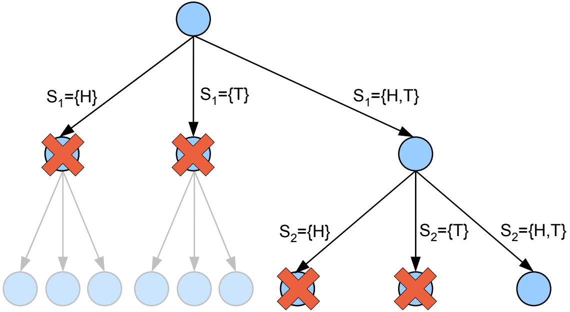
$$\sum_{a_{-1} \in \sigma_{-i}} p(a_{-i}) u_i(a_i, a_{-i}) \leq v_i \qquad \forall i \in \{1,2\}, a_i \notin \sigma_i$$

$$p_i(a_i) \geq 0 \qquad \forall i \in \{1,2\}, a_i \in \sigma_i$$

$$p_i(a_i) = 0 \qquad \forall i \in \{1,2\}, a_i \notin \sigma_i$$

$$\sum_{a_i \in \sigma_i} p_i(a_i) = 1 \qquad \forall i \in \{1,2\}$$

The ideas that make SEM fast



(1) The size of the tree

(2) Dominance

(3) Test Given Support (TGS)