

# Multiunit Auctions

## Lecture 21

# Lecture Overview

- 1 Recap
- 2 Simple Multiunit Auctions
- 3 Unlimited Supply
- 4 General Multiunit Auctions

# Going beyond IPV

- common value model
  - motivation: oil well
  - winner's curse
  - things can be improved by revealing more information
- general model
  - IPV + common value
  - example motivation: private value plus resale

# Risk Attitudes

What kind of auction would the **auctioneer** prefer?

- **Buyer is not risk neutral:**
  - no change under various risk attitudes for second price
  - in first-price, increasing bid amount increases probability of winning, decreases profit. This is good for risk-averse bidder, bad for risk-seeking bidder.
  - Risk averse, IPV: First  $\succ$  [Japanese = English = Second]
  - Risk seeking, IPV: Second  $\succ$  First
- **Auctioneer is not risk neutral:**
  - revenue is fixed in first-price auction (the expected amount of the second-highest bid)
  - revenue varies in second-price auction, with the same expected value
  - thus, a risk-averse seller prefers first-price to second-price.

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  - there are  $k$  identical goods for sale in a single auction
  - every bidder only wants one unit
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- how else can we sell the goods?
  - **pay-your-bid**: “discriminatory” pricing, because bidders will pay different amounts for the same thing
  - **lowest winning bid**: very similar to VCG, but ensures that bidders don't pay zero if there are fewer bids than units for sale
  - **sequential single-good auctions**

# Revenue Equivalence

## Theorem (Revenue equivalence theorem, multiunit version)

*Assume that each of  $n$  risk-neutral agents has an independent private valuation for a single unit of  $k$  identical goods at auction, drawn from a common cumulative distribution  $F(v)$  that is strictly increasing and atomless on  $[\underline{v}, \bar{v}]$ . Then any efficient auction mechanism in which any agent with valuation  $\underline{v}$  has an expected utility of zero yields the same expected revenue, and hence results in any bidder with valuation  $v_i$  making the same expected payment.*

# Sequential Auctions

Although we can apply the revelation principle, for greater intuition we can also use backward induction to derive the equilibrium strategies in finitely-repeated second-price auctions.

- everyone should bid **honestly** in the final auction
- we can also compute a bidder's **expected utility** (conditioned on type) in that auction
- in the second-last auction, bid the difference between valuation and the **expected utility for losing**
  - i.e., bid valuation minus the expected utility for playing the second auction
  - why: consider affine transformation of valuations subtracting this constant expected utility
- combining these last two auctions together, there's some expected utility to playing both of them
- now this is the "expected utility of losing"
- apply **backward induction**

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# Unlimited Supply

- consider MP3 downloads as an example of a multiunit good.
- They differ from the other examples we gave:
  - the seller can produce additional units at zero marginal cost
  - hence has an effectively unlimited supply of the good
  - The seller will not face any supply restrictions other than those she imposes herself.
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- How should such goods be auctioned?
  - the seller will have to artificially reduce supply
  - the *first* unit of the good might have been very expensive to produce!

# Optimal Single Price

If we *knew* bidders' valuations but had to offer the goods at the same price to all bidders, it would be easy to compute the optimal single price.

## Definition (Optimal single price)

The **optimal single price** is calculated as follows.

- 1 Order the bidders in descending order of valuation; let  $v_i$  denote the  $i$ th-highest valuation.
- 2 Calculate  $opt \in \arg \max_{i \in \{1, \dots, n\}} i \cdot v_i$ .
- 3 The optimal single price is  $v_{opt}$ .

# Random Sampling Auction

## Definition (Random sampling optimal price auction)

The *random sampling optimal price auction* is defined as follows.

- 1 Randomly partition the set of bidders  $N$  into two sets,  $N_1$  and  $N_2$  (i.e.,  $N = N_1 \cup N_2$ ;  $N_1 \cap N_2 = \emptyset$ ; each bidder has probability 0.5 of being assigned to each set).
- 2 Using the procedure above find  $p_1$  and  $p_2$ , where  $p_i$  is the optimal single price to charge the set of bidders  $N_i$ .
- 3 Then set the allocation and payment rules as follows:
  - For each bidder  $i \in N_1$ , award a unit of the good if and only if  $b_i \geq p_2$ , and charge the bidder  $p_2$ ;
  - For each bidder  $j \in N_2$ , award a unit of the good if and only if  $b_j \geq p_1$ , and charge the bidder  $p_1$ .



# Results

## Theorem

*Random sampling optimal price auctions are dominant-strategy truthful, weakly budget balanced and ex post individually rational.*

## Theorem

*The random sampling optimal price auction always yields expected revenue that is at least a  $(\frac{1}{4.68})$  constant fraction of the revenue that would be achieved by charging bidders the optimal single price, subject to the constraint that at least two units of the good must be sold.*

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# Multiunit Demand

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How does VCG behave when (some) bidders may want more than a single unit of the good?

- no longer a  $k + 1$ st-price auction
- instead, all winning bidders who won the same number of units will pay the same amount as each other.
  - the change in social welfare from dropping any of these bidders is the same.
- Bidders who win different numbers of units will not necessarily pay the same per unit prices.
- However, bidders who win larger numbers of units will pay at least as much in total (not necessarily per unit) as bidders who won smaller numbers of units
  - their impact on social welfare will always be at least as great

# Winner Determination for Multiunit Demand

- Let  $m$  be the number of units available, and let  $\hat{v}_i(k)$  denote bidder  $i$ 's declared valuation for being awarded  $k$  units.
- It's no longer computationally easy to **identify the winners**—now it's a (NP-complete) weighted knapsack problem:

$$\text{maximize } \sum_{i \in N} \sum_{1 \leq k \leq m} \hat{v}_i(k) x_{k,i} \quad (1)$$

$$\text{subject to } \sum_{i \in N} \sum_{1 \leq k \leq m} k \cdot x_{k,i} \leq m \quad (2)$$

$$\sum_{1 \leq k \leq m} x_{k,i} \leq 1 \quad \forall i \in N \quad (3)$$

$$x_{k,i} = \{0, 1\} \quad \forall 1 \leq k \leq m, i \in N \quad (4)$$

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- $x_{k,i}$  indicates whether bidder  $i$  is allocated exactly  $k$  units
- maximize: sum of agents' valuations for the chosen allocation
- (2): number of units allocated does not exceed number available
- (3): no more than one  $x_{\cdot,i}$  is nonzero for any  $i$
- (4): all  $x$ 's must be integers

# Multiunit Valuations

How can bidders express their valuations in a multiunit auction?

- $m$  homogeneous goods, let  $S$  denote some set
- **general**: let  $p_1, \dots, p_m$  be arbitrary, non-negative real numbers. Then  $v(S) = \sum_{j=1}^{|S|} p_j$ .
- **downward sloping**: general, but  $p_1 \geq p_2 \geq \dots \geq p_m$
- **additive**:  $v(S) = c|S|$
- **single-item**:  $v(S) = c$  if  $s \neq \emptyset$ ; 0 otherwise
- **fixed-budget**:  $v(S) = \min(c|S|, b)$
- **majority**:  $v(S) = c$  if  $|S| \geq m/2$ , 0 otherwise