Survey of the Computational Complexity of Finding a Nash Equilibrium

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1 Abstract

This survey reviews the complexity classes of interest for describing the problem of finding a Nash Equilibrium, the reductions that led to the conclusion that finding a Nash Equilibrium of a game in the normal form is **PPAD**-complete, and the reduction from a succinct game with a nice expected utility function to finding a Nash Equilibrium in a normal form game with two players.

2 Introduction

2.1 Motivation

For many years it was unknown whether a Nash Equilibrium of a normal form game could be found in polynomial time. The motivation for being able to due so is to find game solutions that people are satisfied with. Consider an entertaining game that some friends are playing. If there is a set of strategies that constitutes and equilibrium each person will probably be satisfied. Likewise if financial games such as auctions could be solved in polynomial time in a way that all of the participates were satisfied that they could not do any better given the situation, then that would be great. There has been much progress made in the last fifty years toward finding the complexity of this problem. It turns out to be a hard problem unless **PPAD** = **FP**.

This survey reviews the complexity classes of interest for describing the problem of finding a Nash Equilibrium, the reductions that led to the conclusion that finding a Nash Equilibrium of a game in the normal form is **PPAD**-complete, and the reduction from a succinct game with a nice expected utility function to finding a Nash Equilibrium in a normal form game with two players.

2.2 Definition of Nash Equilibrium

In [2], Nash defines a Nash Equilibrium formally. He also proves that there must exist at least one mixed Nash Equilibrium in any game of two or more players.

A Nash Equilibrium is a strategy profile in which no player can unilaterally change his strategy to achieve a better expected payoff. A variation that is considered in some of the visited works of this survey is the ϵ-Nash Equilibrium. An ϵ-Nash Equilibrium is a strategy profile in which no player can unilaterally change his strategy profile to achieve an expected payoff ϵ greater than his current expected payoff.

A mixed Nash Equilibrium is a strategy profile in which at least one of the players' strategies is mixed. A strategy profile is a tuple of all of the players' strategies. A mixed strategy is a probability distribution over a player's actions.

2.3 Definition of complexity classes

The complexity class of interest for this survey is **PPAD**-complete. Before defining this, this survey includes the definition of related complexity classes.

FNP is the complexity class of function problems that are associated with a language in **NP**. Likewise **FP** is the complexity class of function problems that are associated with a language in **P**. Functions problems are of the form, for a polynomial-time decidable, polynomially balanced relation $R(x, y)$ on L, a language in **NP**, given a string x, find a string y such that $R(x, y)$, where there is a y with $R(x, y)$ if and only if $x \in L$. If no string y exists such that $R(x, y)$, return "no". (Paraphrased from [1].)

TFNP is the complexity class of function problems in **FNP** where R as defined above is a total function. That is, "for every string x there is at least one string y such that $R(x, y)$." [1] **TFNP** is defined in [3].

PPA is the complexity class of search problems defined as "Given x, find a leaf on $G(x)$ other than [the standard leaf]," [4] where "$G(x)$ is a symmetric graph of degree at most two." [4] The standard leaf is defined by the problem to exist in the same position for every graph. The existence of another leaf is proven by the even leaves argument, "In any directed graph with one unbalanced node (node with outdegree different from its indegree), there is another unbalanced node," [8] where an unbalanced node in a graph of degree at most two is a leaf node. This class of problems is contained in **TFNP**. **PPA** is defined in [6] as **PLF**, and in [4] as **PPA**.

PPAD is similar to **PPA**, but $G(x)$ is a directed graph with at most one predecessor and at most one successor per vertex. The problem is to find a sink or a source on a directed graph given the standard source. It is contained in **PPA**. **PPAD** is defined in [6] as **PDLF**, and in [4] as **PPAD**.

3 Reductions

Several papers successively succeed in showing that finding a Nash Equilibrium for normal form games with progressively smaller numbers of players is **PPAD**-complete.

3.1 r-Nash to 4-Nash

In [7] the authors lay groundwork for proving the complexity of finding a Nash Equilibrium for games with fewer numbers of players. The authors work under a hierarchy 2-NASH, 3-NASH, ..., r-NASH, ..., that means finding a Nash Equilibrium for r players, r-NASH, is at least as hard as s-NASH, where $r > s$. This is easy to see as a game with r players can be reduced to a game with $r + 1$ players by adding a dummy player. The authors state that their key contribution in this paper is the collapsing of the hierarchy to the fourth level, that is r-NASH, $r \geq 4$, reduces to 4-NASH. The two reductions of interest in this paper are from d-GRAPHICAL NASH to d^2-NASH and from r-NASH to 3-GRAPHICAL NASH to 4-NASH. d-GRAPHICAL NASH is the problem of finding a Nash Equilibrium on a graphical game where the graph has degree at most d. Graphical games are mentioned in 4 and defined in [12].

The reduction from d-GRAPHICAL NASH to d^2-NASH uses coloring of the graph to assign vertices to players and then has each player pair off and play Matching Pennies to ensure the players are randomizing over their vertices. The Nash Equilibria are preserved under this reduction and there is a 1-to-1 correspondence between a Nash Equilibrium of d-GRAPHICAL NASH and a Nash Equilibrium of d^2-NASH.

The reduction from r-NASH to 3-GRAPHICAL NASH involves creating a graph of vertices representing (player, pair) strategies such that the Nash Equilibria are preserved and there is a 1-to-1 correspondence between a Nash Equilibrium of r-NASH and a Nash Equilibrium of 3-GRAPHICAL NASH. The degree of the graph is three, and each vertex has the strategies $\{0, 1\}$.

3.2 4-Nash is PPAD-complete

In [8] the authors show that finding a Nash Equilibrium for games with four players in hard. The paper shows a reduction from 3-DIMENSIONAL BROUWER to 3-GRAPHICAL NASH to 4-NASH. The reduction from 3-DIMENSIONAL BROUWER to 3-GRAPHICAL NASH is based on devising circuit element-like graphical game gadgets to reconstruct the input circuit to 3-DIMENSIONAL BROUWER as a graphical game with degree three. The gadgets developed in this paper are used (sometimes as modified versions) in [9], [10], [11], and [13]. The composition of these game gadgets are used to compute the value of the Brouwer function. The Brouwer function is "evaluated" to the utilities of the players, and the fixpoints constitute Nash Equilibria. The graphical game can further be reduced to 4-Nash by coloring the undirected graph associated with the graph given in the proof using four colors.

3.3 3-Nash is PPAD-complete

Two papers independently show that 3-NASH is **PPAD**-complete. [9] modifies the arithmetic gadgets from [7] to create a graphical game whose graph can be colored with three colors. [10] uses a different proof with the unmodified gadgets from [7]. Their graph still requires four colors, but uses other features of the graph to prove only three players are required.

3.4 2-Nash is PPAD-complete

In [11] 2-Nash is shown to be **PPAD**-complete. The proof is a reduction from the problem 3-Dimensional Brouwer to 2-Nash. The reduction is similar to that in [8]. A graphical game is constructed from the input circuit to 3-Dimensional Brouwer using arithmetic and logic gadgets with two classes of nodes. Player 1 can strategize over the arithmetic nodes, for which the probability of Player 1 playing the strategy "1" at that node is a real number in an ϵ-Nash equilibrium. Player 2 can strategize over the interior nodes, of which each gadget has exactly one. The gadgets are added to a variant of the two player Matching Pennies game.

The game constructed has size polynomial on the size of the input to 3-Dimensional Brouwer and can be computed in polynomial time on the same input. Given a ϵ-Nash equilibrium of the game constructed, a panchromatic vertex of the input to 3-Dimensional Brouwer will have a form given in Theorem 2 of [11].

4 Succinctly Representable Multiplayer Games

"A normal-form game with n players and s strategies per player requires ns^n integers (utilities) for its representation." [9] Because the size of the normal form is exponential in the number of players, representations with smaller size bounds have been sought in the hopes that they may prove more tractable to work with. Graphical games appear in [12] as undirected graphs of n vertices for n players. Each player has a payoff matrix that depends only on the action of himself and his neighbors. Graphical games were instrumental for proving the reductions presented in the previous section.

A formal definition of succinct games is given in [13]. Some examples according to [13] include graphical games, congestion games, multimatrix games, semi-anonymous games, local effect games, scheduling games, hypergraphical games, network design games, and facility location games. Some examples according to [15] include graphical games, sparse games, symmetric games, anonymous games, extensive form games, congestion games, network congestion games, local effect games, facility location games, and multimatrix games. The definition of succinct games in [13] does not include extensive form games, so some discussion on this point will follow the result. A result that counters the hope that finding a Nash Equilibrium for general succinctly representable games appears in [13].

Theorem [13]. *If for a succinct game G of polynomial type there is a bounded division-free straight-line program of polynomial length for computing* Expected Utility*, then the problem of computing a Nash equilibrium in the succinct game polynomially reduces to the problem of computing a Nash equilibrium of a 2-player game.*

A game has polynomial type when the number of agents and the size of the action sets have a polynomial bound on the size of the representation. A bounded division-free straight-line program of polynomial length can be understood as any binary arithmetic function that can be computed in polynomial time with the operators $+, -, *$ on leaves of the expression that are constant or the input to the function. The problem Expected Utility, as expected, computes the expected utility for player i given a mixed strategy profile.

The proof of this Theorem is involved, but it is executed as a reduction from a succinct game to a computing a Nash Equilibrium on a graphical game, which has a reduction to 2-Nash. The graphical game constructed is similar in flavor to that presented in [8] where nodes represent the probabilities of players playing pure strategies with some bookkeeping nodes to guarantee the Nash Equilibrium of the graphical game is the same as that of the original game. The problem with this approach is that the ϵ'-Nash approximations of the graphical game do not always coincide with an ϵ-Nash of the original game. To deal with this, the authors of [13] use a "trim and renormalize strategy" that ignores all pure strategies whose probabilities are less than a threshold (trim) and renormalizes the remaining probabilities. The threshold is calculated to guarantee the equilibrium of the graphical game gives the equilibrium of the original game.

4.1 Extensive Form Games

Extensive form games are included in [15]'s list of succinct games but not in [13]'s. The reasoning in [15]'s side seems to be that when considering extensive form games, they consider not just perfect information games, but also imperfect information games. [13]'s definition requires the game to have polynomial type, which a perfect information game does not have. It can be argued that an imperfect information game has polynomially bounded action sets if the number of information sets is fixed, but this is really only true in for behavioral strategies rather than for actions.

Regardless, computing the behavioral Nash Equilibrium and the subgame perfect equilibrium in the extensive form is shown in [13] to be polynomially reducible to 2-Nash.

5 Conclusion

This survey reviewed the complexity classes of interest for describing the problem of finding a Nash Equilibrium, **FNP**, **TFNP**, **PPA**, and **PPAD**. Second this survey reviewed the reductions that led to the conclusion that finding a Nash Equilibrium of a game in the normal form is **PPAD**-complete for $2, 3$, and 4 players. Lastly this survey reviewed the reduction from a succinct game with a nice expected utility function to finding a Nash Equilibrium in a normal form game with two players and commented on different uses of "succinct" as applied to extensive form games in the literature. The works cited concluded that finding a Nash Equilibrium is hard unless **PPAD** = **FP**.

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