Budget Constraints Impact on Multi Unit Auctions

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Abstract

We study multi unit auctions in presence of budget constraints. This situation is very common in real world but has not received proper attention in theoretical literature. We will study the impact of budget on the different aspects of multi unit auctions. These results try to define a frame work in which multi unit auctions with budget constraints are working and problem we are facing in designing mechanisms.

1 Introduction

There is always been a gap between the auction literature and auctions in practice. In practice there are always budget constraints an upper bound on the maximum amount of money each agent can spend in an auction. There are few works that study the impact of budget constraints on auctions and even less works on the multi unit auctions although multi unit auctions are becoming very popular in e-commerce.

One of the reasons there have been few works on auctions with budget constraints is that the existence of budget constraints introduce complexity into our model. Budget constraints change our simple neat quasi-linear setting into a non quasi-leaner setting and most of the results and theories are based on the quasi-linear assumption about the agent's utility in auctions. The celebrated VCG mechanism breaks down in this situation and no more guarantees desired properties like incentive compatibility.

There have been few works that consider budget constraints on classical standard auctions like first-price and second-price auctions and try to analyze the impact of budget constraints on outcome of this auctions[2]. The others tried to consider a simpler situation like selling single good to two buyers while maximizing seller's revenues[7]. There are few works which considered multi unit auctions when buyers have budget constraints. These works try to prove some results in this setting and design mechanisms that yield outcomes with specific properties like incentive compatibility and revenue maximization.

2 Preliminaries

In a multi unit auction there is m units of good that auctioneer tries to sell to n agents. Each agent $i \in n$ has a valuation v_i and a budget b_i. The budget constraints our hard meaning that agents can't spend more than their budgets. This constraint is modeled differently in different works in some of the works it is defined as agent's utility will be negative infinity in case he spends more than his budget and in some other works the allocation in which agent receives items that cost more than his budget is defined as infeasible.

We define the utility for agent i when he receives x_i units of the good as follow:
$u_i = v_i x_i - p_i$ when $p_i < b_i$
$u_i = -\infty$ when $p_i > b_i$
p_i is the agent i's payment.

Since revelation principle holds we consider direct mechanisms. In this setting each agent submit a two parameter bid to the mechanism. The first parameter is interpreted as that agent's announced valuation for each unit and the second parameter is interpreted as that agent's announced budget. The mechanism's output defines an allocation vector $(x_1, x_2, \ldots, x_n)$ and a payment vector $(p_1, p_2, \ldots, p_n)$. We show each agent $i's$ bid as (v_i, b_i) and other agents except agent i bids as (v_{-i}, b_{-i}).

In some previous works[8][6] in order to remove the complexity and keep the utilities in quasi-linear setting. They have tried to model the budget as an upper bound on the valuation an agent can get instead of his payment. In this setting valuation for an agent is defined as $v' = min(v, b)$ in this equation v is agent's valuation and b is agent's budget. This model misses the point that budgets are real constraints on the amount of money an agent can spend in a real auction and also mislead us to assume that since we are still in quasi-linear setting the VCG mechanism will keep its promises about the incentive compatibility. Following example shows us that in this situation VCG mechanism is not incentive compatible: assume we have two units of good and two agents and the truthful values for valuation and budget for each agents are given by $(v_1, b_1) = (10, 10)$ and $(v_2, b_2) = (1, 10)$. In this setting the mechanism assumes the valuation for the first agent for one or two unit of the good is 10 and therefor allocates one unit to each agents to maximize the total utility which is $(10 + 1)$. The payment for first agent is 1 and for the second agent is 0 and utility for the first agent is 9. Now consider this situation in which first agent lies about his valuation and announce his valuation and budget as $(5, 10)$ then the mechanism allocate both units to this agent at a price of 2. Thus the first agent will have incentive to lie and achieve a higher utility of 18. This example shows that in this setting the VCG mechanism is not incentive compatible.

Since we are no longer in quasi-linear setting defining the social-welfare is tricky and ambiguous. Consider a situation in which we have only one unit of good and an agent has a very high valuation for a unit of good and 0 budge if we want to maximize social-welfare we must allocate the unit of good to this agent and charge him nothing. It also gives agents incentive to lie to the mechanism. It is certainly a direction for future works to define a proper measure for welfare of the auction under budget constraints.

Before we introduce some of the impossibility results for multi unit auctions with budge constraints we have to define some properties for and auction.

Consumer sovereignty: For every agent i and every bid vector (v_{-i}, b_{-i}) there is a bid (v_i, b_i) such that if agent i bid that bid fixing other agent's bid then agent i receives all units of good.

This property says that each agent can win all the unit of goods if he bids high enough.

Independence of irrelevant alternatives (IIA): For any agent i and a bid vector (v, b) if agent i receives no units of good. The allocation must not change when all the other agents bid the same and agent$i$ bids $(0, 0)$.

This property says if an agent who has not won any units of good leave the auction allocation of the good in the new auction without that agent must not change (it is possible in the new auction agents have to pay different amount of money compared to previous auction).

Individual rationality: An agent's utility from participating in auction is nonnegative.

This property means for each agent i we have $p_i \leq x_i v_i$.

Pareto-optimality: An allocation (x, p) is pareto-optimal if there is no other allocation (x', p') in which all other players including auctioneer are better off. It means there is no (x', p') for which we have ${x_i}' {v_i}' - {p_i}' \geq x_i v_i - p_i$ and $\sum_i {p_i}' \geq \sum_i p_i$ with at least one of the inequalities strict.

Some times in order to model m as a limit toward infinity. We assume that we have a single item that is infinitely divisible.

Pareto-optimality (For infinitely divisible good): An allocation (x, p) is pareto-optimal if and only if (a) $\sum_i x_i = 1$ and (b) for all i such that $x_i > 0$ we have that for all j with $v_j > v_i, p_j = b_j$.

This property means that a player can only get a non-zero allocation only if all players with higher valuations have spent all their budgets.

In the next section we will provide some impossibility results for multi unit auctions with budget constraints.

3 Impossibility results

In this section we provide some results which show that implementing multi unit auctions with certain properties is impossible. The first thing we are interested in is incentive compatibility in every mechanism we design. When agents have private budges and private valuation it has been shown that[3] :

Theorem 1: There is no deterministic incentive compatible auction for two agents and two unit of a good that satisfies consumer sovereignly, IIA and strong non-bundling.

They have defined strong non-bundling for two agents in[3] as follow:

Strong non-bundling: There is always a bid vector (v, b) for two agents in which the mechanism assign to each agent a unit of good when there is only two units of good.

We can generalize this result to the case in which there are more than two agents by assuming all agents except two submit zero bids.

In another work[4] it has been shown that if budgets are publicly known. A mechanism can be designed which is unique and has the properties: incentive compatibility pareto-optimality and always allocates all items.

They have also shown that if budgets are private we can prove following result.

Theorem 2: There is no incentive compatible and pareto-optimal mechanism if the budgets are private.

4 Revenue

Multi unit auctions with budget constraints arise often in practice. Some examples of such auction include privatization in European countries and advertisement on search result pages on websites like Google, Yahoo! and MSN. Considering this huge market the revenue maximization is a critical problem and there are very few works considering this problem in multi unit setting. The first work which tries to address this problem and design and introduce a mechanism for advertisement slots on a search result page is[5].

First we define the optimal single price auction in which we assume that we know all the true valuations and budget constraints for all agents and try to maximize the revenue of the mechanism. The optimal single price revenue of selling m units of good to agents with bid vector (v, b) which is sorted in decreasing order according to valuations of the agents can be defined as follow $F = F(v, b, m) = min(\sum_{j=1}^{k^*} b_j, v_{k^*}m)$ where $k^* = k(v, b, m) = argmin_i(\sum_{j=1}^{i} b_j \geq v_{i+1}m)$.

Let define α as any value such that $\alpha \leq \frac{F}{b_{max}}$ in which b_{max} is the largest budget amongst the agents who has received item in the optimal auction. Clearly α defines an upper bound on the fraction any single agent can contributes to the optimal solution and b_{max} defines an upper bound on the amount any single bidder can contribute to the auction.

Let assume we have a single good which is infinitely divisible or our goods are divisible. Now we define the optimal multi-price auction in which we assume that we know all the true valuations and budget constraints for all agents and try to maximize the revenue of the mechanism. The optimal multi-price auction is defined as: $k^T = argmax_i(\sum_{j=1}^{i-1} \frac{b_j}{v_j} < m)$ and revenue $T(v, b, m) = \sum_{j=1}^{k^T-1} b_j + (m - \sum_{j=1}^{k^T-1} \frac{b_j}{v_j})v_{k^T}$.

Theorem 3: For a multi unit auction with budget constraint we have $T(v, b, m) \leq 2F(v, b, m)$ and this is a tight bound.

For a proof for this theorem you can refer to [1]. Now let define competitive ratio β as follow:

Competitive ratio: We say auction A has competitive ratio β against z if for all bid vectors (v, b) such that $\alpha \geq z$ the expected profit of $A(v, b, m)$ satisfies $\beta \geq \frac{F(v,b,m)}{E[A(v,b,m)]}$.

After defining the competitive ratio we can introduce the following theorem which defines an upper bound on the competitive ratio for any truthful mechanism against optimal multi-price auction.

Theorem 4: For any truthful randomized auction A, the competitive ratio against the optimal multi-price auction when $\alpha \geq 2$ is at least $2 - \epsilon$.

There are several direct truthful mechanisms that try to maximize the revenue. In the rest of this section we introduce some of them. The first mechanism we are going to introduce is a mechanism that is proposed in [3]. This mechanism guarantees to achieves an asymptotically maximized revenue compared to optimal single-price auction.

Asymptotically Mechanism:

- Partition agents randomly into two sets A and B by independently putting each agent into either set uniformly at random with probability $\frac{1}{2}$.
- from the set of valuations v_i of agents $i \in A$, choose p_A to be the price which maximizes the revenue of the selling at most $\frac{m}{2}$ units in A. In other words if the u_i's are sorted in decreasing order for

  $$i_0 = min(i : \sum_{j=1}^{i-1} b_j \geq \frac{v_i m}{2})$$

  define $p_A = v_{i_0} - 1$. Compute p_B in the same manner.
- Consider agents in A in a random order and allocate at most $\frac{m}{2}$ units to them as follow. In every step, if the valuation of agent i satisfies $v_i \geq p_B$ allocate $\frac{b_i}{p_B}$ units to i, or all

remaining units if less than $\frac{b_i}{p_B}$ units remain. Charge i a price of p_B per unit. Apply the same procedure to the set B using the threshold value p_A.

The proof for truthfulness of this mechanism is provided in [3]. In the next theorem we propose some guarantee about the revenue of the above mechanism.

Theorem 5: The mechanism described in previous section is truthful. Furthermore, for all $0 < \delta < 1$ the mechanism has revenue at least $(1-\delta)F$ with probability $1 - O(e^{\frac{-c\delta^2}{\epsilon}})$ for some constant c and $\epsilon = \frac{b_{max}}{F}$.

For the final mechanism we propose a mechanism that does well if the α parameter is known in advance. The argument in defense of this auction is that an auctioneer can gain some information about the valuation and budget of the agents through the time. Thus he can choose the best suitable auction according to α.

Masking Mechanism:

- For every agent i, compute $F_{-i} = F(v_{-i}, b_{-i}, m)$ and set $p_i = \frac{F_{-i}}{m}$.
- For agents i with $v_i > p_i$, each of $\frac{b_i}{p_i}$ is allocated independently with probability $\frac{\alpha-1}{\alpha+1}$ at price p_i per item.
- if $\sum_{i:v_i>p_i} \dfrac{b_i}{p_i} \geq m$, end the auction. Otherwise, allocate $m - \sum_{i:v_i>p_i} \dfrac{b_i}{p_i}$ items to agents with $v_i = p_i$. Allocate arbitrarily, not exceeding budgets, at price p_i per item.

Theorem 6: The masking auction is truthful.

Theorem 7: if $\alpha \geq 5.828$, the making auction has competitive ration $\frac{(\alpha+1)\alpha}{(\alpha-1)^2}$ and never allocates more than $\frac{\alpha+1}{\alpha-1}m$ items.

A comparison between different mechanisms which try to maximize revenue in different setting is presented in [1].

5 Summary and Future Work

Multi unit auctions with budget constraints have gained a great deal of interest in recent years. Especially in e-commerce and Internet advertising there have been few works considering this problem and the few pioneering works try to establish a theoretical frame work for multi unit auctions with budget constraints.

There is still a lot of space in both defining and expanding the frame work by providing new results in this setting like defining the social-welfare function in this situation and designing mechanisms which achieve a better revenues in all the possible situations.

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