Computing Nash Equilibrium; Maxmin

Lecture 5

Lecture Overview

Pareto Optimality

- Idea: sometimes, one outcome o is at least as good for every agent as another outcome o', and there is some agent who strictly prefers o to o'
  - in this case, it seems reasonable to say that o is better than o'
  - we say that o Pareto-dominates o'.
- An outcome o^* is Pareto-optimal if there is no other outcome that Pareto-dominates it.
  - a game can have more than one Pareto-optimal outcome
  - every game has at least one Pareto-optimal outcome

Best Response

- If you knew what everyone else was going to do, it would be easy to pick your own action
- Let $a_{-i} = \langle a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \rangle$.
  - now $a = (a_{-i}, a_i)$

- Best response: $a_i^* \in BR(a_{-i})$ iff $\forall a_i \in A_i,\ u_i(a_i^*, a_{-i}) \geq u_i(a_i, a_{-i})$

Nash Equilibrium

- Now we return to the setting where no agent knows anything about what the others will do
- Idea: look for stable action profiles.
- $a = \langle a_1, \ldots, a_n \rangle$ is a ("pure strategy") Nash equilibrium iff $\forall i,\ a_i \in BR(a_{-i})$.

Mixed Strategies

- It would be a pretty bad idea to play any deterministic strategy in matching pennies
- Idea: confuse the opponent by playing randomly
- Define a strategy s_i for agent i as any probability distribution over the actions A_i.
  - pure strategy: only one action is played with positive probability
  - mixed strategy: more than one action is played with positive probability
    - these actions are called the support of the mixed strategy
- Let the set of all strategies for i be S_i
- Let the set of all strategy profiles be $S = S_1 \times \ldots \times S_n$.

Utility under Mixed Strategies

- What is your payoff if all the players follow mixed strategy profile $s \in S$?
  - We can't just read this number from the game matrix anymore: we won't always end up in the same cell
- Instead, use the idea of expected utility from decision theory:

$$u_i(s) = \sum_{a \in A} u_i(a) Pr(a|s)$$

$$Pr(a|s) = \prod_{j \in N} s_j(a_j)$$

Best Response and Nash Equilibrium

Our definitions of best response and Nash equilibrium generalize from actions to strategies.

- Best response:
  - $s_i^* \in BR(s_{-i})$ iff $\forall s_i \in S_i,\ u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$
- Nash equilibrium:
  - $s = \langle s_1, \ldots, s_n \rangle$ is a Nash equilibrium iff $\forall i,\ s_i \in BR(s_{-i})$
- Every finite game has a Nash equilibrium! [Nash, 1950]
  - e.g., matching pennies: both players play heads/tails 50%/50%

Lecture Overview

Computing Mixed Nash Equilibria: Battle of the Sexes

	B	F
B	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

- It's hard in general to compute Nash equilibria, but it's easy when you can guess the support
- For BoS, let's look for an equilibrium where all actions are part of the support

Computing Mixed Nash Equilibria: Battle of the Sexes

	B	F
B	2, 1	0, 0
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- Let player 2 play B with p, F with $1 - p$.
- If player 1 best-responds with a mixed strategy, player 2 must make him indifferent between F and B (why?)

Computing Mixed Nash Equilibria: Battle of the Sexes

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$$u_1(B) = u_1(F)$$
$$2p + 0(1 - p) = 0p + 1(1 - p)$$
$$p = \frac{1}{3}$$

Computing Mixed Nash Equilibria: Battle of the Sexes

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  - Why is player 1 willing to randomize?

Computing Mixed Nash Equilibria: Battle of the Sexes

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- Likewise, player 1 must randomize to make player 2 indifferent.
  - Why is player 1 willing to randomize?
- Let player 1 play B with q, F with $1 - q$.

$$u_2(B) = u_2(F)$$
$$q + 0(1 - q) = 0q + 2(1 - q)$$
$$q = \frac{2}{3}$$

- Thus the strategies $(\frac{2}{3}, \frac{1}{3})$, $(\frac{1}{3}, \frac{2}{3})$ are a Nash equilibrium.

Interpreting Mixed Strategy Equilibria

What does it mean to play a mixed strategy?

- Randomize to confuse your opponent
  - consider the matching pennies example
- Players randomize when they are uncertain about the other's action
  - consider battle of the sexes
- Mixed strategies are a concise description of what might happen in repeated play: count of pure strategies in the limit
- Mixed strategies describe population dynamics: 2 agents chosen from a population, all having deterministic strategies. MS is the probability of getting an agent who will play one PS or another.

Lecture Overview

Fun Game!

	L	R
T	$80, 40$	$40, 80$
B	$40, 80$	$80, 40$

- Play once as each player, recording the strategy you follow.

Fun Game!

	L	R
T	$320, 40$	$40, 80$
B	$40, 80$	$80, 40$

- Play once as each player, recording the strategy you follow.

Fun Game!

	L	R
T	$44, 40$	$40, 80$
B	$40, 80$	$80, 40$

- Play once as each player, recording the strategy you follow.

Fun Game!

	L	R
T	$80, 40$; $320, 40$; $44, 40$	$40, 80$
B	$40, 80$	$80, 40$

- Play once as each player, recording the strategy you follow.
- What does row player do in equilibrium of this game?

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- What does row player do in equilibrium of this game?
  - row player randomizes 50-50 all the time
  - that's what it takes to make column player indifferent
- What happens when people play this game?
  - with payoff of 320, row player goes up essentially all the time
  - with payoff of 44, row player goes down essentially all the time

Lecture Overview

Maxmin Strategies

- Player i's maxmin strategy is a strategy that maximizes i's worst-case payoff, in the situation where all the other players (whom we denote $-i$) happen to play the strategies which cause the greatest harm to i.
- The maxmin value (or safety level) of the game for player i is that minimum amount of payoff guaranteed by a maxmin strategy.

Definition (Maxmin)

The maxmin strategy for player i is $\arg\max_{s_i} \min_{s_{-i}} u_i(s_1, s_2)$, and the maxmin value for player i is $\max_{s_i} \min_{s_{-i}} u_i(s_1, s_2)$.

- Why would i want to play a maxmin strategy?

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- Why would i want to play a maxmin strategy?
  - a conservative agent maximizing worst-case payoff
  - a paranoid agent who believes everyone is out to get him

Minmax Strategies

- Player i's minmax strategy against player $-i$ in a 2-player game is a strategy that minimizes $-i$'s best-case payoff, and the minmax value for i against $-i$ is payoff.
- Why would i want to play a minmax strategy?

Definition (Minmax, 2-player)

In a two-player game, the minmax strategy for player i against player $-i$ is $\arg\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$, and player $-i$'s minmax value is $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$.

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We can generalize to n players.

Definition (Minmax, n-player)

In an n-player game, the minmax strategy for player i against player $j \neq i$ is i's component of the mixed strategy profile s_{-j} in the expression $\arg\min_{s_{-j}} \max_{s_j} u_j(s_j, s_{-j})$, where $-j$ denotes the set of players other than j. As before, the minmax value for player j is $\min_{s_{-j}} \max_{s_j} u_j(s_j, s_{-j})$.

Minmax Theorem

Theorem (Minimax theorem (von Neumann, 1928))

In any finite, two-player, zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and his minmax value.

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2. For both players, the set of maxmin strategies coincides with the set of minmax strategies.

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1. Each player's maxmin value is equal to his minmax value. By convention, the maxmin value for player 1 is called the value of the game.
2. For both players, the set of maxmin strategies coincides with the set of minmax strategies.
3. Any maxmin strategy profile (or, equivalently, minmax strategy profile) is a Nash equilibrium. Furthermore, these are all the Nash equilibria. Consequently, all Nash equilibria have the same payoff vector (namely, those in which player 1 gets the value of the game).