

# VCG

## Lecture 16

# Lecture Overview

- 1 Recap
- 2 VCG
- 3 VCG example
- 4 Individual Rationality
- 5 Budget Balance

# Truthfulness

## Definition (Truthfulness)

A quasilinear mechanism is **truthful** if it is direct and  $\forall i \forall v_i$ , agent  $i$ 's equilibrium strategy is to adopt the strategy  $\hat{v}_i = v_i$ .

- Our definition before, adapted for the quasilinear setting

# Efficiency

## Definition (Efficiency)

A quasilinear mechanism is **strictly Pareto efficient**, or just **efficient**, if in equilibrium it selects a choice  $x$  such that

$$\forall v \forall x', \sum_i v_i(x) \geq \sum_i v_i(x').$$

- An efficient mechanism selects the choice that maximizes the sum of agents' utilities, disregarding monetary payments.
- Called **economic efficiency** to distinguish from other (e.g., computational) notions
- Also called **social-welfare maximization**
- Note: defined in terms of true (not declared) valuations.

# Budget Balance

## Definition (Budget balance)

A quasilinear mechanism is **budget balanced** when

$$\forall v, \sum_i p_i(s(v)) = 0,$$

where  $s$  is the equilibrium strategy profile.

- regardless of the agents' types, the mechanism collects and disburses the same amount of money from and to the agents
- we can also define **weak** or **ex ante** variants

# Individual-Rationality

## Definition (*Ex interim* individual rationality)

A mechanism is **ex interim individual rational** when

$\forall i \forall v_i, \mathbb{E}_{v_{-i}|v_i} v_i(\chi(s_i(v_i), s_{-i}(v_{-i}))) - p_i(s_i(v_i), s_{-i}(v_{-i})) \geq 0$ ,  
where  $s$  is the equilibrium strategy profile.

- no agent loses by participating in the mechanism.
- *ex interim* because it holds for every possible valuation for agent  $i$ , but averages over the possible valuations of the other agents.

## Definition (*Ex post* individual rationality)

A mechanism is **ex post individual rational** when

$\forall i \forall v, v_i(\chi(s(v))) - p_i(s(v)) \geq 0$ , where  $s$  is the equilibrium strategy profile.

# Tractability

## Definition (Tractability)

A quasilinear mechanism is **tractable** when  $\forall a \in A$ ,  $\chi(a)$  and  $p(a)$  can be computed in polynomial time.

- The mechanism is computationally feasible.

# Revenue Maximization

We can also add an objective function to our mechanism. One example: revenue maximization.

## Definition (Revenue maximization)

A mechanism is **revenue maximizing** when, among the set of functions  $\chi$  and  $p$  that satisfy the other constraints, the mechanism selects the  $\chi$  and  $p$  that maximize  $\mathbb{E}_\theta \sum_i p_i(s(\theta))$ , where  $s(\theta)$  denotes the agents' equilibrium strategy profile.

- The mechanism designer can choose among mechanisms that satisfy the desired constraints by adding an objective function such as revenue maximization.



# Revenue Minimization

- The mechanism may not be intended to make money.
- Budget balance may be impossible to satisfy.
- Set weak budget balance as a constraint and add the following objective.

## Definition (Revenue minimization)

A quasilinear mechanism is **revenue minimizing** when, among the set of functions  $\chi$  and  $p$  that satisfy the other constraints, the mechanism selects the  $\chi$  and  $p$  that minimize  $\max_v \sum_i p_i(s(v))$  in equilibrium, where  $s(v)$  denotes the agents' equilibrium strategy profile.

- Note: this considers the **worst case** over valuations; we could consider average case instead.

# Fairness

- **Maxmin fairness**: make the least-happy agent the happiest.

## Definition (Maxmin fairness)

A quasilinear mechanism is **maxmin fair** when, among the set of functions  $\chi$  and  $p$  that satisfy the other constraints, the mechanism selects the  $\chi$  and  $p$  that maximize

$$\mathbb{E}_v \left[ \min_{i \in N} v_i(\chi(s(v))) - p_i(s(v)) \right],$$

where  $s(v)$  denotes the agents' equilibrium strategy profile.

# Price of Anarchy Minimization

- When an efficient mechanism is impossible, we may want to get as close as possible
- Minimize the **worst-case ratio** between optimal social welfare and the social welfare achieved by the given mechanism.

## Definition (Price-of-anarchy minimization)

A quasilinear mechanism **minimizes the price of anarchy** when, among the set of functions  $\chi$  and  $p$  that satisfy the other constraints, the mechanism selects the  $\chi$  and  $p$  that minimize

$$\max_{v \in V} \frac{\max_{x \in X} \sum_{i \in N} v_i(x)}{\sum_{i \in N} v_i(\chi(s(v)))},$$

where  $s(v)$  denotes the agents' equilibrium strategy profile in the *worst* equilibrium of the mechanism—i.e., the one in which  $\sum_{i \in N} v_i(\chi(s(v)))$  is the smallest.

# The Groves Mechanism

## Definition (Groves mechanism)

The **Groves mechanism** is a direct quasilinear mechanism  $(\chi, p)$ , where

$$\chi(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x)$$
$$p_i(\hat{v}) = h_i(\hat{v}_{-i}) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$$

# Groves Properties

## Theorem

*Truth telling is a dominant strategy under the Groves mechanism.*

## Theorem (Green–Laffont)

An *efficient* social choice function  $C : \mathbb{R}^{X^n} \rightarrow X \times \mathbb{R}^n$  can be implemented in dominant strategies for agents with unrestricted quasilinear utilities *only if*  $p_i(v) = h(v_{-i}) - \sum_{j \neq i} v_j(x(v))$ .

# Lecture Overview

- 1 Recap
- 2 VCG**
- 3 VCG example
- 4 Individual Rationality
- 5 Budget Balance

# Clarke Tax

## Definition (Clarke tax)

The **Clarke tax** sets the  $h_i$  term in a Groves mechanism as

$$h_i(\hat{v}_{-i}) = \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}_{-i})).$$

## Definition (Vickrey-Clarke-Groves (VCG) mechanism)

The **Vickrey-Clarke-Groves mechanism** is a direct quasilinear mechanism  $(\chi, p)$ , where

$$\chi(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x)$$

$$p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}_{-i})) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$$

# VCG discussion

$$\chi(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x)$$
$$p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}_{-i})) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$$

- You get paid everyone's utility under the allocation that is actually chosen
  - except your own, but you get that directly as utility
- Then you get charged everyone's utility in the world where you don't participate
- Thus you pay your **social cost**



# VCG discussion

$$\chi(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x)$$

$$p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}_{-i})) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$$

Questions:

- who pays 0?

# VCG discussion

$$\chi(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x)$$
$$p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}_{-i})) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$$

Questions:

- who pays 0?
  - agents who don't affect the outcome

# VCG discussion

$$\chi(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x)$$
$$p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}_{-i})) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$$

Questions:

- who pays 0?
  - agents who don't affect the outcome
- who pays more than 0?

# VCG discussion

$$\chi(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x)$$
$$p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}_{-i})) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$$

Questions:

- who pays 0?
  - agents who don't affect the outcome
- who pays more than 0?
  - (pivotal) agents who make things worse for others by existing

# VCG discussion

$$\chi(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x)$$

$$p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}_{-i})) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$$

Questions:

- who pays 0?
  - agents who don't affect the outcome
- who pays more than 0?
  - (pivotal) agents who make things worse for others by existing
- who gets paid?

# VCG discussion

$$\chi(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x)$$
$$p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}_{-i})) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$$

Questions:

- who pays 0?
  - agents who don't affect the outcome
- who pays more than 0?
  - (pivotal) agents who make things worse for others by existing
- who gets paid?
  - (pivotal) agents who make things better for others by existing

# VCG properties

$$\chi(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x)$$
$$p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}_{-i})) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$$

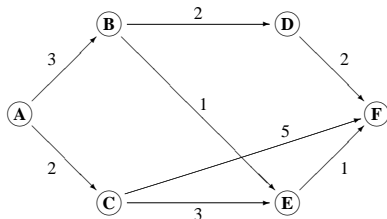
- Because only **pivotal** agents have to pay, VCG is also called the **pivot mechanism**
- It's dominant-strategy truthful, because it's a Groves mechanism

# Lecture Overview

- 1 Recap
- 2 VCG
- 3 VCG example**
- 4 Individual Rationality
- 5 Budget Balance

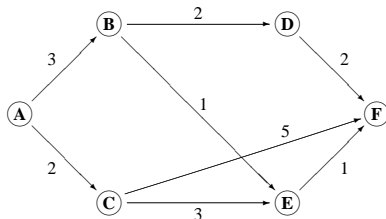


# Selfish routing example



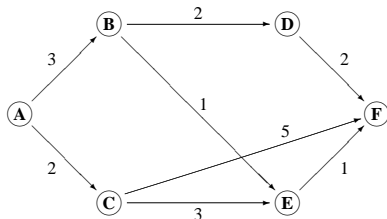
- What outcome will be selected by  $\chi$ ?

# Selfish routing example



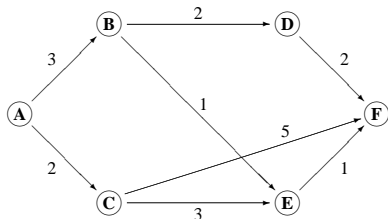
- What outcome will be selected by  $\chi$ ? path *ABEF*.

# Selfish routing example



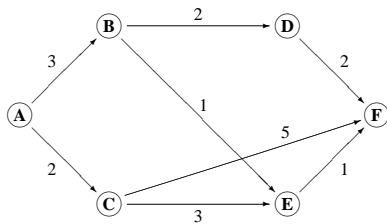
- What outcome will be selected by  $\chi$ ? path  $ABEF$ .
- How much will  $AC$  have to pay?

# Selfish routing example



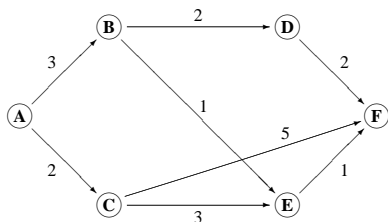
- What outcome will be selected by  $\chi$ ? path  $ABEF$ .
- How much will  $AC$  have to pay?
  - The shortest path taking his declaration into account has a length of 5, and imposes a cost of  $-5$  on agents other than him (because it does not involve him). Likewise, the shortest path without  $AC$ 's declaration also has a length of 5. Thus, his payment  $p_{AC} = (-5) - (-5) = 0$ .
  - This is what we expect, since  $AC$  is not pivotal.
  - Likewise,  $BD$ ,  $CE$ ,  $CF$  and  $DF$  will all pay zero.

# Selfish routing example



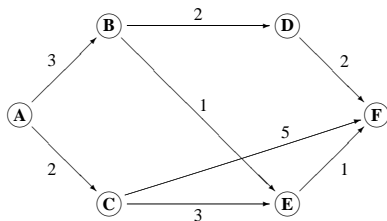
- How much will  $AB$  pay?

# Selfish routing example



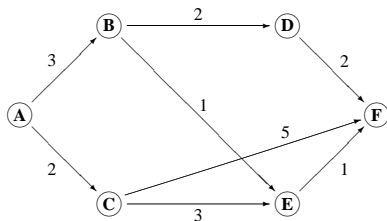
- How much will  $AB$  pay?
  - The shortest path taking  $AB$ 's declaration into account has a length of 5, and imposes a cost of 2 on other agents.
  - The shortest path without  $AB$  is  $ACEF$ , which has a cost of 6.
  - Thus  $p_{AB} = (-6) - (-2) = -4$ .

# Selfish routing example



- How much will  $BE$  pay?

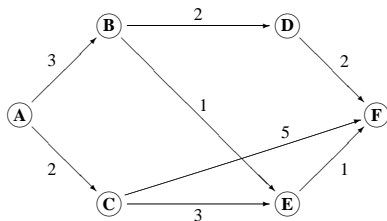
# Selfish routing example



- How much will  $BE$  pay?  $p_{BE} = (-6) - (-4) = -2$ .

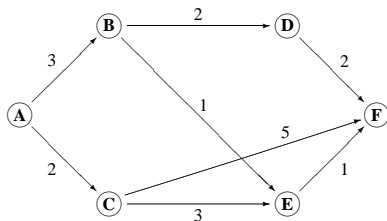


# Selfish routing example



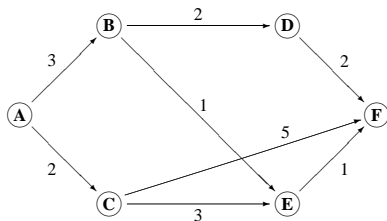
- How much will  $BE$  pay?  $p_{BE} = (-6) - (-4) = -2$ .
- How much will  $EF$  pay?

# Selfish routing example



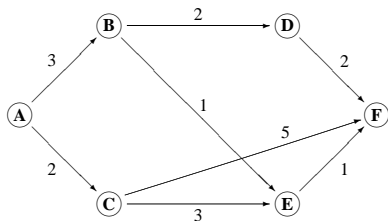
- How much will  $BE$  pay?  $p_{BE} = (-6) - (-4) = -2$ .
- How much will  $EF$  pay?  $p_{EF} = (-7) - (-4) = -3$ .

# Selfish routing example



- How much will  $BE$  pay?  $p_{BE} = (-6) - (-4) = -2$ .
- How much will  $EF$  pay?  $p_{EF} = (-7) - (-4) = -3$ .
  - $EF$  and  $BE$  have the same costs but are paid different amounts. Why?

# Selfish routing example



- How much will  $BE$  pay?  $p_{BE} = (-6) - (-4) = -2$ .
- How much will  $EF$  pay?  $p_{EF} = (-7) - (-4) = -3$ .
  - $EF$  and  $BE$  have the same costs but are paid different amounts. Why?
  - $EF$  has more *market power*: for the other agents, the situation without  $EF$  is worse than the situation without  $BE$ .

# Lecture Overview

- 1 Recap
- 2 VCG
- 3 VCG example
- 4 Individual Rationality**
- 5 Budget Balance

## Two definitions

### Definition (Choice-set monotonicity)

An environment exhibits **choice-set monotonicity** if  $\forall i, X_{-i} \subseteq X$ .

- removing any agent weakly decreases—that is, never increases—the mechanism's set of possible choices  $X$

### Definition (No negative externalities)

An environment exhibits **no negative externalities** if

$$\forall i \forall x \in X_{-i}, v_i(x) \geq 0.$$

- every agent has zero or positive utility for any choice that can be made without his participation

# Example: road referendum

## Example

Consider the problem of holding a referendum to decide whether or not to build a road.

- The set of choices is independent of the number of agents, satisfying choice-set monotonicity.
- No agent negatively values the project, though some might value the situation in which the project is not undertaken more highly than the situation in which it is.

# Example: simple exchange

## Example

Consider a market setting consisting of agents interested in buying a single unit of a good such as a share of stock, and another set of agents interested in selling a single unit of this good. The choices in this environment are sets of buyer-seller pairings (prices are imposed through the payment function).

- If a new agent is introduced into the market, no previously-existing pairings become infeasible, but new ones become possible; thus choice-set monotonicity is satisfied.
- Because agents have zero utility both for choices that involve trades between other agents and no trades at all, there are no negative externalities.



# VCG Individual Rationality

## Theorem

*The VCG mechanism is ex-post individual rational when the choice set monotonicity and no negative externalities properties hold.*

## Proof.

All agents truthfully declare their valuations in equilibrium. Then

$$\begin{aligned} u_i &= v_i(\chi(v)) - \left( \sum_{j \neq i} v_j(\chi(v_{-i})) - \sum_{j \neq i} v_j(\chi(v)) \right) \\ &= \sum_i v_i(\chi(v)) - \sum_{j \neq i} v_j(\chi(v_{-i})) \end{aligned} \quad (1)$$

$\chi(v)$  is the outcome that maximizes social welfare, and that this optimization could have picked  $\chi(v_{-i})$  instead (by choice set monotonicity). Thus,

$$\sum_j v_j(\chi(v)) \geq \sum_j v_j(\chi(v_{-i})).$$

# VCG Individual Rationality

## Theorem

*The VCG mechanism is ex-post individual rational when the choice set monotonicity and no negative externalities properties hold.*

## Proof.

$$\sum_j v_j(\chi(v)) \geq \sum_j v_j(\chi(v_{-i})).$$

Furthermore, from no negative externalities,

$$v_i(\chi(v_{-i})) \geq 0.$$

Therefore,

$$\sum_i v_i(\chi(v)) \geq \sum_{j \neq i} v_j(\chi(v_{-i})),$$

and thus Equation (1) is non-negative.

# Lecture Overview

- 1 Recap
- 2 VCG
- 3 VCG example
- 4 Individual Rationality
- 5 Budget Balance**

# Another property

## Definition (No single-agent effect)

An environment exhibits **no single-agent effect** if  $\forall i, \forall v_{-i}, \forall x \in \arg \max_y \sum_j v_j(y)$  there exists a choice  $x'$  that is feasible without  $i$  and that has  $\sum_{j \neq i} v_j(x') \geq \sum_{j \neq i} v_j(x)$ .

## Example

Consider a single-sided auction. Dropping an agent just reduces the amount of competition, making the others better off.

# Good news

## Theorem

*The VCG mechanism is weakly budget-balanced when the no single-agent effect property holds.*

## Proof.

Assume truth-telling in equilibrium. We must show that the sum of transfers from agents to the center is greater than or equal to zero.

$$\sum_i p_i(v) = \sum_i \left( \sum_{j \neq i} v_j(\chi(v_{-i})) - \sum_{j \neq i} v_j(\chi(v)) \right)$$

From the no single-agent effect condition we have that

$$\forall i \sum_{j \neq i} v_j(\chi(v_{-i})) \geq \sum_{j \neq i} v_j(\chi(v)).$$

Thus the result follows directly.

# More good news

## Theorem (Krishna & Perry, 1998)

*In any Bayesian game setting in which VCG is ex post individually rational, VCG collects at least as much revenue as any other efficient and ex interim individually-rational mechanism.*

- This is somewhat surprising: does not require dominant strategies, and hence compares VCG to all Bayes–Nash mechanisms.
- A useful corollary: VCG is as budget balanced as any efficient mechanism can be
  - it satisfies weak budget balance in every case where *any* dominant strategy, efficient and *ex interim* IR mechanism would be able to do so.

# Bad news

## Theorem (Green–Laffont; Hurwicz)

*No dominant-strategy incentive-compatible mechanism is always both efficient and weakly budget balanced, even if agents are restricted to the simple exchange setting.*

## Theorem (Myerson–Satterthwaite)

*No Bayes-Nash incentive-compatible mechanism is always simultaneously efficient, weakly budget balanced and ex-interim individual rational, even if agents are restricted to quasilinear utility functions.*