Consolidation of Unorganized Point Clouds for Surface Reconstruction

Supplementary Material III: Convergence argument on improved locally optimal projection

To verify the contracting behavior of LOP iteration with new repulsion term, we first reshape the point sets X and P as column vectors $\mathbf{x}$ and $\mathbf{p}$ and write the optimization problem as $\min_{\mathbf{x}} F(\mathbf{x}, \mathbf{x}^k)$, where

$$\begin{aligned} F(\mathbf{x}, \mathbf{c}) \equiv & \sum_{i \in I} \sum_{j \in J} \|A_i^T \mathbf{x} - B_j^T \mathbf{p}\| \theta(\|A_i^T \mathbf{c} - B_j^T \mathbf{p}\|) \\ & + \sum_{i \in I} \lambda_i \sum_{i' \in I \setminus \{i\}} \eta(\|A_i^T \mathbf{x} - A_{i'}^T \mathbf{c}\|) \theta(\|L_{ii'}^T \mathbf{c}\|). \end{aligned} \tag{1}$$

Here, A_i, B_j and $L_{ii'} = A_i - A_{i'}$ are appropriate structured matrices. Then, from the necessary condition for a minimizer in (1), $F'(\mathbf{x}^{k+1}, \mathbf{x}^k) = \mathbf{0}$, we derive the fixed point iteration expression

$$\begin{aligned} \mathbf{x}^{k+1} &= \sum_{i \in I} A_i \sum_{j \in J} \frac{\alpha_{ij}^k B_j^T \mathbf{p}}{\sum_{j \in J} \alpha_{ij}^k} + \sum_{i \in I} \frac{\lambda_i A_i}{\sum_{j \in J} \alpha_{ij}^k} \sum_{i' \in I \setminus \{i\}} \beta_{ii'}^k L_{ii'}^T \mathbf{x}^k \\ &= \sum_{i \in I} A_i \sum_{j \in J} \frac{\alpha_{ij}^k B_j^T \mathbf{p}}{\sum_{j \in J} \alpha_{ij}^k} + \mu \sum_{i \in I} A_i \sum_{i' \in I \setminus \{i\}} \frac{\beta_{ii'}^k L_{ii'}^T \mathbf{x}^k}{\sum_{i' \in I \setminus \{i\}} \beta_{ii'}^k}, \end{aligned}$$

where $\lambda_i = \mu \frac{\sum_{j \in J} \alpha_{ij}^k}{\sum_{i' \in I \setminus \{i\}} \beta_{ii'}^k} > 0$, $\alpha_{ij}^k = \frac{\theta(\|A_i^T \mathbf{x}^k - B_j^T \mathbf{p}\|)}{\|A_i^T \mathbf{x}^k - B_j^T \mathbf{p}\|} > 0$, and $\beta_{ii'}^k = \frac{\theta(\|L_{ii'}^T \mathbf{x}^k\|)|\eta'(\|L_{ii'}^T \mathbf{x}^k\|)|}{\|L_{ii'}^T \mathbf{x}^k\|} > 0$.

Next, assuming that there exists a solution $\mathbf{x}^*$ for the fixed point iteration such that $F'(\mathbf{x}^*, \mathbf{x}^*) = \mathbf{0}$, we have

$$\begin{aligned} F'(\mathbf{x}^*, \mathbf{x}^k) &= F'(\mathbf{x}^*, \mathbf{x}^k) - F'(\mathbf{x}^*, \mathbf{x}^*) \approx J_1(\mathbf{x}^k - \mathbf{x}^*), \\ F'(\mathbf{x}^*, \mathbf{x}^k) &= F'(\mathbf{x}^*, \mathbf{x}^k) - F'(\mathbf{x}^{k+1}, \mathbf{x}^k) \approx J_2(\mathbf{x}^* - \mathbf{x}^{k+1}), \end{aligned}$$

where $J_1 = \frac{\partial F'}{\partial \mathbf{x}^k}$ and $J_2 = \frac{\partial F'}{\partial \mathbf{x}^{k+1}}$. Thus,

$$\mathbf{x}^{k+1} - \mathbf{x}^* \approx -J_2^{-1} J_1 (\mathbf{x}^k - \mathbf{x}^*).$$

The fixed point iteration is therefore contracting, indicating convergence of the LOP fixed point iteration provided that $\mathbf{x}^0$ is already sufficiently close to $\mathbf{x}^*$, if the corresponding spectral radius ρ (i.e., the maximum modulus of the eigenvalues) satisfies $\rho(-J_2^{-1} J_1(\mathbf{x}^k)) < 1$.

Now, some tedious but otherwise straightforward algebra shows that, if $\eta' = -1$, then for any complex-valued vector $\mathbf{y} \neq \mathbf{0}$, we have the inner product

$$\begin{aligned} & \langle \mathbf{y}, -J_2^{-1} J_1(\mathbf{x}^k) \mathbf{y} \rangle \\ = & \sum_{i \in I} \sum_{j \in J} \frac{\alpha_{ij}^k}{\widehat{\alpha}_i^k} \left(\frac{\langle A_i^T \mathbf{x}^k - B_j^T \mathbf{p}, A_i^T \mathbf{y} \rangle^2}{\|A_i^T \mathbf{x}^k - B_j^T \mathbf{p}\|^2} \right) \\ - \mu & \sum_{i \in I} \sum_{i' \in I \setminus \{i\}} \frac{\beta_{ii'}^k}{\widehat{\beta}_i^k} \frac{\langle L_{ii'}^T \mathbf{x}^k, L_{ii'}^T \mathbf{y} \rangle \langle L_{ii'}^T \mathbf{x}^k, A_i^T \mathbf{y} \rangle}{\|L_{ii'}^T \mathbf{x}^k\|^2}, \end{aligned}$$

where $\widehat{\alpha}_i^k = \sum_{j \in J} \alpha_{ij}^k$ and $\widehat{\beta}_i^k = \sum_{i' \in I \setminus \{i\}} \beta_{ii'}^k$. Application of the Cauchy-Schwarz inequality yields the bounds

$$\begin{aligned} 0 < & \sum_{i \in I} \sum_{j \in J} \frac{\alpha_{ij}^k}{\widehat{\alpha}_i^k} \left(\frac{\langle A_i^T \mathbf{x}^k - B_j^T \mathbf{p}, A_i^T \mathbf{y} \rangle^2}{\|A_i^T \mathbf{x}^k - B_j^T \mathbf{p}\|^2} \right) \\ & \leq \sum_{i \in I} \sum_{j \in J} \frac{\alpha_{ij}^k}{\widehat{\alpha}_i^k} \|A_i^T \mathbf{y}\|^2 \leq \|\mathbf{y}\|^2, \\ 0 < & \sum_{i \in I} \sum_{i' \in I \setminus \{i\}} \frac{\beta_{ii'}^k}{\widehat{\beta}_i^k} \frac{\langle L_{ii'}^T \mathbf{x}^k, A_i^T \mathbf{y} - A_{i'}^T \mathbf{y} \rangle \langle L_{ii'}^T \mathbf{x}^k, A_i^T \mathbf{y} \rangle}{\|L_{ii'}^T \mathbf{x}^k\|^2} \\ & \leq 2\|\mathbf{y}\|^2. \end{aligned}$$

Recalling also that $\mu \in [0, 1/2)$, the magnitude of the Rayleigh quotient is bounded as

$$0 < \frac{|\langle \mathbf{y}, -J_2^{-1} J_1 \mathbf{y} \rangle|}{\|\mathbf{y}\|^2} < 1, \quad \forall\, \mathbf{y} \neq \mathbf{0}.$$

Thus, we have the spectral radius $\rho(-J_2^{-1} J_1) < 1$, and so the linearization of the fixed point iteration is contracting. Such a result is not available for the original LOP algorithm.

The convergence argument given above for LOP with the new repulsion $\eta(r) = -r$ and uniform weights cannot be easily adapted in the general weighted LOP (WLOP) setting. However, empirically, we have always obtained error plots that are indicative of convergence; see Figure 5(b) in the paper for example. An interactive comparison of convergence behaviors between LOP and WLOP can be found in the supplementary video.