CPSC 303, Fall Term, 2010
Assignment 4, due Monday November 15th

Please show all your work: e-mail your MATLAB programs to `cs303@ugrad.cs.ubc.ca`, but provide a hardcopy of the rest of the assignment. When e-mailing your programs, include your name and student ID in the message's title.

Do not e-mail a complete assignment.

1. Let us denote $x_{\pm 1} = x_0 \pm h$ and $f(x_i) = f_i$. It is known that the difference formula

   $$fpp_0 = (f_1 - 2f_0 + f_{-1})/h^2$$

   provides a 2nd order method for approximating the second derivative of f at x_0, and also that roundoff error increases like h^{-2}.

   Write a MATLAB script using default floating point arithmetic to calculate and plot the actual total error for approximating $f''(1.2)$, with $f(x) = \sin(x)$. Plot the error on a loglog scale for $h = 10^{-k}$, `k = -8:.5:0`. Observe the roughly V shape of the plot and explain it. What is (approximately) the observed optimal h?

2. Continuing with the notation of Question 1, one could define

   $$g_{1/2} = (f_1 - f_0)/h, \quad \text{and} \quad g_{-1/2} = (f_0 - f_{-1})/h.$$

   These approximate to 2nd order the first derivative values $f'(x_0 + h/2)$ and $f'(x_0 - h/2)$, respectively. Then define

   $$fpp_0 = (g_{1/2} - g_{-1/2})/h.$$

   All three derivative approximations here are centered (hence 2nd order), and they are applied to first derivatives hence have roundoff error increasing proportionally to h^{-1}, not h^{-2}. Can we manage to (partially) cheat the hangman in this way?!

(a) Show that in exact arithmetic fpp_0 defined above and in Question 1 are one and the same.

(b) Implement this method and compare to the results of Question 1. Explain your observations.

3. The basic trapezoidal rule for approximating $I_f = \int_a^b f(x)dx$ is based on linear interpolation of f at $x_0 = a$ and $x_1 = b$. The Simpson rule is likewise based on quadratic polynomial interpolation. Consider now a cubic Hermite polynomial, interpolating both f and its derivative f' at a and b. The osculating interpolation formula gives

$$p_3(x) = f(a) + f'(a)(x - a) + f[a, a, b](x - a)^2 + f[a, a, b, b](x - a)^2(x - b),$$

and integrating this yields (after some algebra)

$$I \approx \int_a^b p_3(x)dx = \frac{b-a}{2}[f(a) + f(b)] + \frac{(b-a)^2}{12}[f'(a) - f'(b)].$$

This formula is called the **corrected trapezoidal rule**.

a) Show that the error in the basic corrected trapezoidal rule can be estimated by

$$E(f) = \frac{f''''(\eta)}{720}(b - a)^5.$$

b) Use the basic corrected trapezoidal rule to evaluate approximations for $\int_0^1 e^x dx$ and $\int_{0.9}^1 e^x dx$. Compare errors to those of Example 15.3 in the notes. What are your observations?

4. Continuing Question 3:

a) Derive the **composite corrected trapezoidal rule**. How does it relate to the composite trapezoidal rule? Use both composite trapezoidal and corrected composite trapezoidal to evaluate approximations for $I_f = \int_0^1 e^{-x^2} dx$ with $r = 10$ subintervals. What are your observations? [The exact value is $I_f = 0.746824133..$]

b) Show that the error in the (uncorrected) composite trapezoidal formula can be written as

$$E(f) = I_f - I_{tr} = K_1 h^2 + \mathcal{O}(h^4),$$

where K_1 is independent of h.