Lecture Stat 302
Introduction to Probability - Slides 17

AD

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Jointly Distributed Random Variables

- Assume we have two r.v. X and Y, then we define the joint c.d.f.

$$F(a,b) = P(X \leq a, Y \leq b)$$

- The c.d.f of X is

$$\begin{aligned} F_X(a) &= P(X \leq a, Y \leq \infty) \\ &= \lim_{b \to \infty} P(X \leq a, Y \leq b) \\ &= \lim_{b \to \infty} F(a,b) \end{aligned}$$

- Similarly we have that the c.d.f. of Y is

$$F_Y(b) = P(X \leq \infty, Y \leq b) = \lim_{a \to \infty} F(a,b)$$

Jointly Distributed Random Variables

- Consider

$$
\begin{aligned}
P(X > a, Y > b) &= 1 - P(\{X > a, Y > b\}^c) \\
&= 1 - P(\{X > a, Y > b\}^c) \\
&= 1 - P(\{X > a\}^c \cup \{Y > b\}^c) \\
&= 1 - P(\{X \leq a\} \cup \{Y \leq b\}) \\
&= 1 - [P(X \leq a) + P(Y \leq b) - P(X \leq a, Y \leq b)] \\
&= 1 - F_X(a) - F_X(b) + F(a, b).
\end{aligned}
$$

- For discrete variables, we work directly with the joint pmf

$$p(x, y) = P(X = x, Y = y)$$

from which we obtain

$$p_X(x) = \sum_y p(x, y), \quad p_Y(y) = \sum_x p(x, y).$$

Joint Probability Density Function

- If both X and Y are jointly continuous, then their joint p.d.f. is a non-negative function $f(x, y)$ such that for any set C

$$P\{(X, Y) \in C\} = \int \int_{(x,y) \in C} f(x, y)\, dxdy.$$

- In particular, we have

$$F(a, b) = P(X \leq a, Y \leq b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f(x, y)\, dxdy.$$

- Hence, when we differentiate, we obtain

$$f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y).$$

Marginal Probabily Density Functions

- The marginal pdf f_X of the r.v. X is defined as

$$F(a) = P(X \leq a) = \int_{-\infty}^{a} f_X(x)\, dx$$

- We have

$$P(X \leq a) = P(X \leq a, Y \leq \infty) = \int_{-\infty}^{a} \int_{-\infty}^{\infty} f(x, y)\, dxdy$$

so clearly

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy.$$

- Similarly, we have

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx.$$

Example

- Consider the joint p.d.f.

$$f(x,y) = \begin{cases} kx & \text{for } 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant. Find k?

- We have $\int_0^1 \int_0^1 f(x,y)\,dxdy = 1$ where

$$\begin{aligned} \int_0^1 \int_0^1 f(x,y)\,dxdy &= \int_0^1 \int_0^1 kxdydx = \int_0^1 kxdx \\ &= \left[k\frac{x^2}{2}\right]_0^1 = \frac{k}{2} = 1 \end{aligned}$$

thus $k = 2$.

Example: System Failure

- A device contains two components. The device fails if either component fails. The joint p.d.f. of the lifetimes of the components, measured in hours, is $f(s,t)$ where $0 < s < 1$ and $0 < t < 1$. What is the probability that the device fails during the first half hour of operation?
- We have

$$\begin{aligned}
\text{Proba} &= P\left[(S \leq 0.5) \cup (T \leq 0.5)\right] \\
&= \int_0^{0.5} \int_{0.5}^{1} f(s,t)\, dsdt + \int_0^{1} \int_0^{1/2} f(s,t)\, dsdt
\end{aligned}$$

as this follows directly from a graph.

Example: System Failure

- A device runs until either of two components fails, at which point the device stops running. The joint density function of the lifetimes of the two components, both measured in hours, is
$$f(x,y)=\frac{x+y}{8} \text{ for } 0<x<2 \text{ and } 0<y<2.$$
What is the probability that the device fails during its first hour of operation?
- We have
$$\begin{aligned}\text{Prob.} &= P(\{X\leq 1\}\cup\{Y\leq 1\}) \\ &= P([\{X>1\}\cap\{Y>1\}]^c) \\ &= 1-P([\{X>1\}\cap\{Y>1\}])\end{aligned}$$
where
$$P([\{X>1\}\cap\{Y>1\}])=\int_1^2\int_1^2\frac{x+y}{8}dxdy.$$

Example: System Failure

- We have

$$\begin{aligned}\frac{1}{8}\int_1^2\int_1^2 (x+y)\,dxdy &= \frac{1}{8}\int_1^2\left[\frac{(x+y)^2}{2}\right]_1^2 dy \\ &= \frac{1}{18}\int_1^2\left[(y+2)^2-(y+1)\right]^2 dy \\ &= \frac{1}{18}\left(\left[\frac{(y+2)^3}{3}\right]_1^2-\left[\frac{(y+1)^3}{3}\right]_1^2\right)=\frac{18}{48}\end{aligned}$$

- Hence we have Prob.$=1-\frac{18}{48}=\frac{30}{48}=0.625$.

Example: Insurance Company

- An insurance company insures a large number of drivers. Let X be the r.v. representing the company's losses under collision insurance, and let Y represent the company's losses under liability insurance. X and Y have a joint p.d.f

$$f(x,y) = \begin{cases} \frac{2x+2-y}{4} & \text{for } 0 < x < 1 \text{ and } 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

What is the probability that the total loss is at least 1?

- We need to compute

$$\begin{aligned} P(X+Y>1) &= \int\int_{(x,y):x+y>1} f(x,y)\,dxdy \\ &= \int_0^1 \left[\int_{1-x}^2 f(x,y)\,dy\right] dx \\ &= \int_0^1 \left[\left(\frac{x+1}{2}\right) y - \frac{y^2}{8}\right]_{1-x}^2 dx \end{aligned}$$

Example: Insurance Company

- We have

$$
\begin{aligned}
P(X+Y>1) &= \int_0^1 \left[\frac{5}{8}x^2 + \frac{3}{4}x + \frac{1}{8}\right] dx \\
&= \int_0^1 \left(\frac{5}{8}x^2 + \frac{3}{4}x + \frac{1}{8}\right) dx \\
&= \left[\frac{5}{24}x^3 + \frac{3}{8}x^2 + \frac{x}{8}\right]_0^1 \\
&= \frac{5}{24} + \frac{3}{8} + \frac{1}{8} = \frac{17}{24}.
\end{aligned}
$$