Chapter 7

Problems

33. (a) $E[X^2 + 4X + 4] = E[X^2] + 4E[X] + 4 = \text{Var}(X) + E^2[X] + 4E[X] + 4 = 14$

(b) $\text{Var}(4 + 3X) = \text{Var}(3X) = 9\text{Var}(X) = 45$

2. (a) $6 \cdot 6 \cdot 9 = 324$

(b) $X = (6 - S)(6 - W)(9 - R)$

(c) $$\begin{aligned} E[X] = {} & 6(6)(6)P\{S = 0, W = 0, R = 3\} + 6(3)(9)P\{S = 0, W = 3, R = 0\} \\ & + 3(6)(9)P\{S = 3, W = 0, R = 0\} + 6(5)(7)P\{S = 0, W = 1, R = 2\} \\ & + 5(6)(7)P\{S = 1, W = 0, R = 2\} + 6(4)(8)P\{S = 0, W = 2, R = 1\} \\ & + 4(6)(8)P\{S = 2, W = 0, R = 1\} + 5(4)(9)P\{S = 1, W = 2, R = 0\} \\ & + 4(5)(9)P\{S = 2, W = 1, R = 0\} + 5(5)(8)P\{S = 1, W = 1, R = 1\} \end{aligned}$$

$$= \frac{1}{\binom{21}{3}}\left[216\binom{9}{3} + 324\binom{6}{3} + 420 \cdot 6\binom{9}{2} + 384\binom{6}{2}9 + 360\binom{6}{2}6 + 200(6)(6)(9)\right]$$

$$\approx 198.8$$

3. $E[|X - Y|^a] = \int_0^1\int_0^1 |x - y|^a dydx$. Now

$$\begin{aligned} \int_0^1 |x - y|^a dy &= \int_0^x (x - y)^a dy + \int_x^1 (y - x)^a dy \\ &= \int_0^x u^a du + \int_0^{1-x} u^a du \\ &= [x^{a+1} + (1 - x)^{a+1}]/(a + 1) \end{aligned}$$

Hence,

$$\begin{aligned} E[|X - Y|^a] &= \frac{1}{a+1}\int_0^1 [x^{a+1} + (1 - x)^{a+1}]dx \\ &= \frac{2}{(a+1)(a+2)} \end{aligned}$$

37. Let W_i, $i = 1, 2$, denote the i^{th} outcome.

$$\begin{aligned}\text{Cov}(X, Y) &= \text{Cov}(W_1 + W_2, W_1 - W_2)\\ &= \text{Cov}(W_1, W_1) - \text{Cov}(W_2, W_2)\\ &= \text{Var}(W_1) - \text{Var}(W_2) = 0\end{aligned}$$

38. $$E[XY] = \int_0^\infty \int_0^x y 2e^{-2x} dy dx$$

$$= \int_0^\infty x^2 e^{-2x} dx = \frac{1}{8}\int_0^\infty y^2 e^{-y} dy = \frac{\Gamma(3)}{8} = \frac{1}{4}$$

45. (a) $$\frac{\text{Cov}(X_1 + X_2, X_2 + X_3)}{\sqrt{\text{Var}(X_1 + X_2)}\sqrt{\text{Var}(X_2 + X_3)}} = \frac{1}{2}$$

(b) 0

49. Let C_i be the event that coin i is being flipped (where coin 1 is the one having head probability .4), and let T be the event that 2 of the first 3 flips land on heads. Then

$$P(C_1 \mid T) = \frac{P(T|C_1)P(C_1)}{P(T|C_1)P(C_1) + P(T|C_2)P(C_2)}$$

$$= \frac{3(.4)^2(.6)}{3(.4)^2(.6) + 3(.7)^2(.3)} = .395$$

Now, with N_j equal to the number of heads in the final j flips, we have

$$E[N_{10} \mid T] = 2 + E[N_7 \mid T]$$

Conditioning on which coin is being used, gives

$$E[N_7 \mid T] = E[N_7 \mid TC_1]P(C_1T) + E[N_7TC_2]P(C_2 \mid T) = 2.8(.395) + 4.9(.605) = 4.0705$$

Thus, $E[N_{10} \mid T] = 6.0705$.

51. $$f_{X|Y}(x \mid y) = \frac{e^{-y}/y}{\int_0^y e^{-y}/y\, dx} = \frac{1}{y}, \quad 0 < x < y$$

$$E[X^3 \mid Y = y] = \int_0^y x^3 \frac{1}{y} dx = y^3/4$$

53. Let X denote the number of days until the prisoner is free, and let I denote the initial door chosen. Then

$$\begin{aligned}E[X] &= E[X|I = 1](.5) + E[X|I = 2](.3) + E[X|I = 3](.2)\\ &= (2 + E[X])(.5) + (4 + E[X])(.3) + .2\end{aligned}$$

Therefore,

$$E[X] = 12$$

55. Let N denote the number of ducks. Given $N = n$, let $I_1, \ldots, I_n$ be such that

$$I_i = \begin{cases} 1 & \text{if duck } i \text{ is hit} \\ 0 & \text{otherwise} \end{cases}$$

$$E[\text{Number hit} \mid N = n] = E\left[\sum_{i=1}^{n} I_i\right]$$

$$= \sum_{i=1}^{n} E[I_i] = n\left[1 - \left(1 - \frac{.6}{n}\right)^{10}\right], \text{ since given}$$

$N = n$, each hunter will independently hit duck i with probability $.6/n$.

$$E[\text{Number hit}] = \sum_{n=0}^{\infty} n\left(1 - \frac{.6}{n}\right)^{10} e^{-6}6^n/n!$$

68. (a) $.6e^{-2} + .4e^{-3}$

(b) $.6e^{-2}\frac{2^3}{3!} + .4e^{-3}\frac{3^3}{3!}$

(c) $P\{3 \mid 0\} = \frac{P\{3,0\}}{P\{0\}} = \frac{.6e^{-2}e^{-2}\frac{2^3}{3!} + .4e^{-3}e^{-3}\frac{3^3}{3!}}{.6e^{-2} + .4e^{-3}}$

69. (a) $\int_0^{\infty} e^{-x}e^{-x}dx = \frac{1}{2}$

(b) $\int_0^{\infty} e^{-x}\frac{x^3}{3!}e^{-x}dx = \frac{1}{96}\int_0^{\infty} e^{-y}y^3dy = \frac{\Gamma(4)}{96} = \frac{1}{16}$

(c) $\frac{\int_0^{\infty} e^{-x}e^{-x}\frac{x^3}{3!}e^{-x}dx}{\int_0^{\infty} e^{-x}e^{-x}dx} = \frac{2}{3^4} = \frac{2}{81}$

73. (a) $E[R] = E[E[R \mid S]] = E[S] = \mu$

(b) $\text{Var}(R \mid S) = 1$, $E[R \mid S] = S$
$\text{Var}(R) = 1 + \text{Var}(S) = 1 + \sigma^2$

(c) $f_R(r) = \int f_S(s)F_{R|S}(r \mid s)ds$

$$= C\int e^{-(s-\mu)^2/2\sigma^2}e^{-(r-s)^2/2}ds$$

$$= K\int \exp\left\{-\left(S - \frac{\mu + r\sigma^2}{1+\sigma^2}\right) \Big/ 2\left(\frac{\sigma^2}{1+\sigma^2}\right)\right\} ds \exp\{-(ar^2 + br)\}$$

Hence, R is normal.

(d) $E[RS] = E[E[RS \mid S]] = E[SE[R \mid S]] = E[S^2] = \mu^2 + \sigma^2$

$\text{Cov}(R, S) = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$

75. X is Poisson with mean $\lambda = 2$ and Y is Binomial with parameters 10, 3/4. Hence

(a) $P\{X + Y = 2\} = P\{X = 0\}P\{Y = 2\} + P\{X = 1\}P\{Y = 1\} + P\{X = 2\}P\{Y = 0\}$
$= e^{-2}\binom{10}{2}(3/4)^2(1/4)^8 + 2e^{-2}\binom{10}{1}(3/4)(1/4)^9 + 2e^{-2}(1/4)^{10}$

(b) $P\{XY = 0\} = P\{X = 0\} + P\{Y = 0\} - P\{X = Y = 0\}$
$= e^{-2} + (1/4)^{10} - e^{-2}(1/4)^{10}$

(c) $E[XY] = E[X]E[Y] = 2 \cdot 10 \cdot \frac{3}{4} = 15$

Chapter 7

Theoretical Exercises

1. Let $\mu = E[X]$. Then for any a

$$
\begin{aligned}
E[(X-a)^2 &= E[(X-\mu+\mu-a)^2] \\
&= E[(X-\mu)^2] + (\mu-a)^2 + 2E[(x-\mu)(\mu-a)] \\
&= E[(X-\mu)^2] + (\mu-a)^2 + 2(\mu-a)E[(X-\mu)] \\
&= E[(X-\mu)^2 + (\mu-a)^2
\end{aligned}
$$

2. $$
\begin{aligned}
E[\,|X-a|\,] &= \int_{x<a}(a-x)f(x)dx + \int_{x>a}(x-a)f(x)dx \\
&= aF(a) - \int_{x<a}xf(x)dx + \int_{x>a}xf(x)dx - a[1-F(a)]
\end{aligned}
$$

Differentiating the above yields
derivative = $2af(a) + 2F(a) - af(a) - af(a) - 1$
Setting equal to 0 yields that $2F(a) = 1$ which establishes the result.

17. $\text{Var}\big(\lambda X_1 + (1-\lambda)X_2\big) = \lambda^2\sigma_1^2 + (1-\lambda)^2\sigma_2^2$

$$\frac{d}{d\lambda}(\quad) = 2\lambda\sigma_1^2 - 2(1-\lambda)\sigma_2^2 = 0 \Rightarrow \lambda = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$$

As $\text{Var}(\lambda X_1 + (1-\lambda)X_2) = E\left[(\lambda X_1 + (1-\lambda)X_2 - \mu)^2\right]$ we want this value to be small.

19. $$
\begin{aligned}
\text{Cov}(X+Y, X-Y) &= \text{Cov}(X, X) + \text{Cov}(X, -Y) + \text{Cov}(Y, X) + \text{Cov}(Y, -Y) \\
&= \text{Var}(X) - \text{Cov}(X, Y) + \text{Cov}(Y, X) - \text{Var}(Y) \\
&= \text{Var}(X) - \text{Var}(Y) = 0.
\end{aligned}
$$

32. $E[X_1X_2 \mid Y=y] = E[X_1 \mid Y=y]E[X_2 \mid Y=y] = y^2$

Therefore, $E[X_1X_2 \mid Y] = Y^2$. As $E[X_i \mid Y] = Y$, this gives that

$$E[X_1X_2] = E[E[X_1X_2 \mid Y]] = Ei[Y^2], \quad E[X_i] = E[E[X_i \mid Y]] = E[Y]$$

Consequently,

$$\text{Cov}(X_1, X_2) = E[X_1X_2] - E[X_1]E[X_2] = \text{Var}(Y)$$