Chapter 6 :

Solutions for Problems

2. (a) $p(0, 0) = \dfrac{8\cdot 7}{13\cdot 12} = 14/39$,

$p(0, 1) = p(1, 0) = \dfrac{8\cdot 5}{13\cdot 12} = 10/39$

$p(1, 1) = \dfrac{5\cdot 4}{13\cdot 12} = 5/39$

(b) $p(0, 0, 0) = \dfrac{8\cdot 7\cdot 6}{13\cdot 12\cdot 11} = 28/143$

$p(0, 0, 1) = p(0, 1, 0) = p(1, 0, 0) = \dfrac{8\cdot 7\cdot 5}{13\cdot 12\cdot 11} = 70/429$

$p(0, 1, 1) = p(1, 0, 1) = p(1, 1, 0) = \dfrac{8\cdot 5\cdot 4}{13\cdot 12\cdot 11} = 40/429$

$p(1, 1, 1) = \dfrac{5\cdot 4\cdot 3}{13\cdot 12\cdot 11} = 5/143$

4. (a) $p(0, 0) = (8/13)^2, p(0, 1) = p(1, 0) = (5/13)(8/13),\ p(1, 1) = (5/13)^2$

(b) $p(0, 0, 0) = (8/13)^3$

$p(i, j, k) = (8/13)^2(5/13)$ if $i + j + k = 1$

$p(i, j, k) = (8/13)(5/13)^2$ if $i + j + k = 2$

8. $$f_Y(y) = c\int_{-y}^{y}(y^2 - x^2)e^{-y}dx$$

$$= \frac{4}{3}cy^3e^{-y},\ -0 < y < \infty$$

$\int_0^\infty f_Y(y)dy = 1 \Rightarrow c = 1/8$ and so $f_Y(y) = \dfrac{y^3e^{-y}}{6},\ 0 < y < \infty$

$$f_X(x) = \frac{1}{8}\int_{|x|}^{\infty}(y^2 - x^2)e^{-y}dy$$

$= \dfrac{1}{4}e^{-|x|}(1 + |x|)$ upon using $-\int y^2e^{-y} = y^2e^{-y} + 2ye^{-y} + 2e^{-y}$

9. (b) $f_X(x) = \dfrac{6}{7}\displaystyle\int_0^2\left(x^2 + \frac{xy}{2}\right)dy = \frac{6}{7}(2x^2 + x)$

(c) $P\{X > Y\} = \dfrac{6}{7}\displaystyle\int_0^1\int_0^x\left(x^2 + \frac{xy}{2}dydx\right) = \frac{15}{56}$

(d) $P\{Y > 1/2 \mid X < 1/2\} = P\{Y > 1/2, X < 1/2\}/P\{X < 1/2\}$

$$= \frac{\displaystyle\int_{1/2}^{2}\int_0^{1/2}\left(x^2 + \frac{xy}{2}dxdy\right)}{\displaystyle\int_0^{1/2}(2x^2 + x)dx}$$

14. Let X and Y denoted respectively the locations of the ambulance and the accident of the moment the accident occurs.

$$P\{|Y-X| < a\} = P\{Y < X < Y + a\} + P\{X < Y < X + a\}$$

$$= \frac{2}{L^2}\int_0^L \int_y^{\min(y+a,L)} dxdy$$

$$= \frac{2}{L^2}\left[\int_0^{L-a}\int_y^{y+a} dxdy + \int_{L-a}^{L}\int_y^{L} dxdy\right]$$

$$= 1 - \frac{L-a}{L} + \frac{a}{L^2}(L-a) = \frac{a}{L}\left(2 - \frac{a}{L}\right),\quad 0 < a < L$$

19. $\int_0^1\int_0^x \frac{1}{x}\, dydx = \int_0^1 dx = 1$

(a) $\int_y^1 \frac{1}{x} dx = -\ln(y)$, $0 < y < 1$

(b) $\int_0^x \frac{1}{x} dy = 1$, $0 < y < 1$

(c) $\frac{1}{2}$

(d) Integrating by parts gives that

$$\int_0^1 y\ \ln(y)dy = -1 - \int_0^1 (y\ln(y) - y)dy$$

yielding the result

$$E[Y] = -\int_0^1 y\ln(y)dy = 1/4$$

20. (a) yes: $f_X(x) = xe^{-x}, f_Y(y) = e^{-y}$, $0 < x < \infty$, $0 < y < \infty$

(b) no: $f_X(x) = \int_x^1 f(x,y)dy = 2(1-x)$, $0 < x < 1$

$$f_Y(y) = \int_0^y f(x,y)dx = 2y,\ 0 < y < 1$$

22. (a) No, since the joint density does not factor.

(b) $f_X(x) = \int_0^1 (x+y)dy = x + 1/2$, $0 < x < 1$.

(c) $P\{X + Y < 1\} = \int_0^1\int_0^{1-x} (x+y)dydx$

$$= \int_0^1 [x(1-x) + (1-x)^2/2]dx = 1/3$$

27. (a) $P\{X + Y \le a\} = \int_0^a\int_0^{a-x} e^{-y}dydx = a - 1 + e^{-a}$, $a < 1$

$$= \int_0^1\int_0^{a-x} e^{-y}dydx = 1 - e^{-a}(e-1),\ a > 1$$

(b) $P\{Y > X/a\} = \int_0^1\int_{x/a}^{\infty} e^{-y}dydx = a(1 - e^{-1/a})$

33. Let X denote Jill's score and let Y be Jack's score. Also, let Z denote a standard normal random variable.

(a)
$$\begin{aligned} P\{Y > X\} &= P\{Y - X > 0\} \\ &\approx P\{Y - X > .5\} \\ &= P\left\{\frac{Y - X - (160-170)}{\sqrt{(20)^2 + (15)^2}} > \frac{.5 - (160-170)}{\sqrt{(20)^2+(15)^2}}\right\} \\ &\approx P\{Z > .42\} \approx .3372 \end{aligned}$$

(b)
$$\begin{aligned} P\{X + Y > 350\} &= P\{X + Y > 350.5\} \\ &= P\left\{\frac{X + Y - 330}{\sqrt{(20)^2 + (15)^2}} > \frac{20.5}{\sqrt{(20)^2+(15)^2}}\right\} \\ &\approx P\{Z > .82\} \approx .2061 \end{aligned}$$

34. Let X and Y denote, respectively, the number of males and females in the sample that never eat breakfast. Since

$$E[X] = 50.4,\ \mathrm{Var}(X) = 37.6992,\ \ E[Y] = 47.2,\ \mathrm{Var}(Y) = 36.0608$$

it follows from the normal approximation to the binomial that is approximately distributed as a normal random variable with mean 50.4 and variance 37.6992, and that Y is approximately distributed as a normal random variable with mean 47.2 and variance 36.0608. Let Z be a standard normal random variable.

(a) $$P\{X+Y \geq 110\} = P\{X+Y \geq 109.5\} = P\left\{\frac{X+Y-97.6}{\sqrt{73.76}} \geq \frac{109.5-97.6}{\sqrt{73.76}}\right\} \approx P\{Z > 1.3856\} \approx .0829$$

(b) $$P\{Y \geq X\} = P\{Y - X \geq -.5\} = P\left\{\frac{Y-X-(-3.2)}{\sqrt{73.76}} \geq \frac{-.5-(-3.2)}{\sqrt{73.76}}\right\} \approx P\{Z \geq .3144\} \approx .3766$$

35. (a) $P\{X_1 = 1 \mid X_2 = 1\} = 4/12 = 1 - P\{X_1 = 0 \mid X_2 = 1\}$

(b) $P\{X_1 = 1 \mid X_2 = 0\} = 5/12 = 1 - P\{X_1 = 0 \mid X_2 = 0\}$

36. (a) $P\{X_1 = 1 \mid X_2 = 1\} = 5/13 = 1 - P\{X_1 = 0 \mid X_2 = 1\}$

(b) same as in (a)

39. (a) $P\{X = j, Y = i\} = \dfrac{1}{5}\dfrac{1}{j}, j = 1, \ldots, j,\ i = 1, \ldots, j$

(b) $P\{X = j \mid Y = i\} = \dfrac{1}{5j}\Big/\sum_{k=i}^{5} 1/5\,k = \dfrac{1}{j}\Big/\sum_{k=i}^{5} 1/k,\ 5 \geq j \geq i.$

(c) No.

42. (a) $f_{X|Y}(x \mid y) = \dfrac{xe^{-x(y+1)}}{\int xe^{-x(y+1)}dx} = (y+1)^2 xe^{-x(y+1)},\ 0 < x$

(b) $f_{Y|X}(y \mid x) = \dfrac{xe^{-x(y+1)}}{\int xe^{-x(y+1)}dy} = xe^{-xy},\ 0 < y$

$$P\{XY < a\} = \int_0^{\infty}\int_0^{a/x} xe^{-x(y+1)}dydx = \int_0^{\infty}(1 - e^{-a})e^{-x}dx = 1 - e^{-a}$$

$f_{XY}(a) = e^{-a},\ 0 < a$

5. (a) For $a > 0$

Chapter 6 :

Solutions for Theoretical Exercises

$$F_Z(a) = P\{X \le aY\}$$
$$= \int_0^{\infty}\int_0^{a/y} f_X(x) f_Y(y)\,dx\,dy$$
$$= \int_0^{\infty} F_X(ay) f_Y(y)\,dy$$
$$f_Z(a) = \int_0^{\infty} f_X(ay)\, y f_Y(y)\,dy$$

(b)
$$F_Z(a) = P\{XY < a\}$$
$$= \int_0^{\infty}\int_0^{a/y} f_X(x) f_Y(y)\,dx\,dy$$
$$= \int_0^{\infty} F_X(a/y) f_Y(y)\,dy$$
$$f_Z(a) = \int_0^{\infty} f_X(a/y)\frac{1}{y} f_Y(y)\,dy$$

If X is exponential with rate λ and Y is exponential with rate μ then (a) and (b) reduce to

(a) $F_Z(a) = \int_0^{\lambda} \lambda e^{-\lambda a y}\, y \mu e^{-\mu y}\,dy$

(b) $F_Z(a) = \int_0^{\infty} \lambda e^{-\lambda a/y}\frac{1}{y}\mu e^{-\mu y}\,dy$

9. $P\{\min(X_1, \ldots, X_n) > t\} = P\{X_1 > t, \ldots, X_n > t\}$
$= e^{-\lambda t} \ldots e^{-\lambda t} = e^{-n\lambda t}$

thus showing that the minimum is exponential with rate $n\lambda$.

7. (a) $P\{cX \le a\} = P\{X \le a/c\}$ and differentiation yields

$$f_{cX}(a) = \frac{1}{c} f_X(a/c) = \frac{\lambda}{c} e^{-\lambda a/c} (\lambda a/c)^{t-1} \Gamma(t).$$

Hence, cX is gamma with parameters $(t, \lambda/c)$.

16. $P(X = n, Y = m) = \sum_i P(X = n, Y = m | X_2 = i) P(X_2 = i)$

$$= e^{-(\lambda_1+\lambda_2+\lambda_3)} \sum_{i=0}^{\min(n,m)} \frac{\lambda_1^{n-i}}{(n-1)!}\frac{\lambda_3^{m-i}}{(m-i)!}\frac{\lambda_2^{i}}{i!}$$

18. $P\{U > s \mid U > a\} = P\{U > s\}/P\{U > a\}$
$= \dfrac{1-s}{1-a}, \; a < s < 1$

$P\{U < s \mid U < a\} = P\{U < s\}/P\{U < a\}$
$= s/a, \; 0 < s < a$

Hence, $U \mid U > a$ is uniform on $(a, 1)$, whereas $U \mid U < a$ is uniform over $(0, a)$.