Chapter 4

Problems: Q7,8,13,17,20,21,25,38,40,45,48,52,53,55,60,64,71,74,79
Theoretical Exercises: Q14,20,27,32

Problem

7 a) max value to appear in the two rolls - {1, 2, 3, 4, 5, 6}

b) min. value to appear in the two rolls : {1, 2, 3, 4, 5, 6}

c) sum of the two rolls : {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

d) the value of the 1st roll minus the value of the 2nd roll

{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5}

8. (a) $p(6) = 1 - (5/6)^2 = 11/36, p(5) = 2\ 1/6\ \ 4/6 + (1/6)^2 = 9/36$
$p(4) = 2\ 1/6\ \ 3/6 + (1/6)^2 = 7/36, p(3) = 2\ 1/6\ \ 2/6 + (1/6)^2 = 5/36$
$p(2) = 2\ 1/6\ \ 1/6 + (1/6)^2 = 3/36, p(1) = 1/36$

(d) $p(5) = 1/36, p(4) = 2/36, p(3) = 3/36, p(2) = 4/36, p(1) = 5/36$
$p(0) = 6/36, p(-j) = p(j), j > 0$

13. $p(0) = P$\{no sale on first and no sale on second\}
$= (.7)(.4) = .28$
$p(500) = P$\{1 sale and it is for standard\}
$= P$\{1 sale\}/2
$=[P$\{sale, no sale\} + P\{no sale, sale\}]/2
$= [(.3)(.4) + (.7)(.6)]/2 = .27$

$p(1000) = P$\{2 standard sales\} + P\{1 sale for deluxe\}
$= (.3)(.6)(1/4) + P$\{1 sale\}/2
$= .045 + .27 = .315$

$p(1500) = P$\{2 sales, one deluxe and one standard\}
$= (.3)(.6)(1/2) = .09$

$p(2000) = P$\{2 sales, both deluxe\} $= (.3)(.6)(1/4) = .045$

20. (a) $P\{x > 0\} = P$\{win first bet\} + P\{lose, win, win\}
$= 18/38 + (20/38)(18/38)^2 \approx .5918$

(b) No, because if the gambler wins then he or she wins \$1.
However, a loss would either be \$1 or \$3.

(c) $E[X] = 1[18/38 + (20/38)(18/38)^2] - [(20/38)2(20/38)(18/38)] - 3(20/38)^3 \approx -.108$

Problem

17)

Recall: $P(X<b) = P(\lim_{n\to\infty}\{X \le b-\frac{1}{n}\})$

$= \lim_{n\to\infty} P\{X \le b-\frac{1}{n}\}$

a). $P(X=1) = P(X\le 1) - P(X<1) = F(1) - \lim_{n\to\infty} F(1-\frac{1}{n})$

$= \left(\frac{1}{2}+\frac{1-1}{4}\right) - \lim_{n\to\infty}\left(\frac{1-\frac{1}{n}}{4}\right)$

$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$ #

$P(X=2) = P(X\le 2) - P(X<2) = \frac{1}{6}$ #

$P(X=3) = P(X\le 3) - P(X<3) = \frac{1}{12}$ #

b) $P(\frac{1}{2}<X<\frac{3}{2}) = P(X<\frac{3}{2}) - P(X\le\frac{1}{2})$

$= \lim_{n\to\infty}\left[\frac{1}{2}+\frac{\frac{3}{2}-\frac{1}{n}-1}{4}\right] - \frac{\frac{1}{2}}{4} = \frac{1}{2}$ #

21. (a) $E[X]$ since whereas the bus driver selected is equally likely to be from any of the 4 buses, the student selected is more likely to have come from a bus carrying a large number of students.

(b) $P\{X=i\} = i/148$, $i = 40, 33, 25, 50$

$E[X] = [(40)^2 + (33)^2 + (25)^2 + (50)^2]/148 \approx 39.28$
$E[Y] = (40 + 33 + 25 + 50)/4 = 37$

25. (a) $\frac{1}{10}(1+2+\ldots+10) = \frac{11}{2}$

(b) after 2 questions, there are 3 remaining possibilities with probability 3/5 and 2 with probability 2/5. Hence.

$$E[\text{Number}] = \frac{2}{5}(3) + \frac{3}{5}\left[2+\frac{1}{3}+2\frac{2}{3}\right] = \frac{17}{5}.$$

The above assumes that when 3 remain, you choose 1 of the 3 and ask if that is the one.

38. (a) $E[(2+X)^2] = \text{Var}(2+X) + (E[2+X])^2 = \text{Var}(X) + 9 = 14$

(b) $\text{Var}(4+3X) = 9\,\text{Var}(X) = 45$

40. $\binom{5}{4}(1/3)^4(2/3)^1 + (1/3)^5 = 11/243$

45. with 3: $P\{\text{pass}\} = \frac{1}{3}\left[\binom{3}{2}(.8)^2(.2)+(.8)^3\right]+\frac{2}{3}\left[\binom{3}{2}(.4)^2(.6)+(.4)^3\right]$

$= .533$

with 5: $P\{\text{pass}\} = \frac{1}{3}\sum_{i=3}^{5}\binom{5}{i}(.8)^i(.2)^{5-i}+\frac{2}{3}\sum_{i=3}^{5}\binom{5}{i}(.4)^i(.6)^{5-i}$

$= .3038$

48. The probability that a package will be returned is $p = 1 - (.99)^{10} - 10(.99)^9(.01)$. Hence, if someone buys 3 packages then the probability they will return exactly 1 is $3p(1-p)^2$.

52. (a) $1 - e^{-3.5} - 3.5e^{-3.5} = 1 - 4.5e^{-3.5}$

(b) $4.5e^{-3.5}$

Since each flight has a small probability of crashing it seems reasonable to suppose that the number of crashes is approximately Poisson distributed.

53. (a) The probability that an arbitrary couple were both born on April 30 is, assuming independence and an equal chance of having being born on any given date, $(1/365)^2$. Hence, the number of such couples is approximately Poisson with mean $80{,}000/(365)^2 \approx .6$. Therefore, the probability that at least one pair were both born on this date is approximately $1 - e^{-.6}$.

(b) The probability that an arbitrary couple were born on the same day of the year is 1/365. Hence, the number of such couples is approximately Poisson with mean $80{,}000/365 \approx 219.18$. Hence, the probability of at least one such pair is $1 - e^{-219.18} \approx 1$.

55. $\frac{1}{2}e^{-3}+\frac{1}{2}e^{-4.2}$

60. $P\{\text{beneficial} \mid 2\} = \dfrac{P\{2|\text{beneficial}\}3/4}{P\{2|\text{ beneficial}\}3/4 + P\{2|\text{ not beneficial}\}1/4}$

$= \dfrac{e^{-3}\frac{3^2}{2}\frac{3}{4}}{e^{-3}\frac{3^2}{2}\frac{3}{4}+e^{-5}\frac{5^2}{2}\frac{1}{4}}$

64. (a) $1 - \sum_{i=0}^{7} e^{-4}4^i/i! \equiv p$

(b) $1 - (1-p)^{12} - 12p(1-p)^{11}$

(c) $(1-p)^{i-1}p$

71. (a) $\left(\frac{26}{38}\right)^5$

(b) $\left(\frac{26}{38}\right)^3 \frac{12}{38}$

74. (a) $\left(\frac{2}{3}\right)^5$

(b) $\binom{8}{5}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^3 + \binom{8}{6}\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^2 + \binom{8}{7}\left(\frac{2}{3}\right)^7\left(\frac{1}{3}\right) + \left(\frac{2}{3}\right)^8$

(c) $\binom{5}{4}\left(\frac{2}{3}\right)^5 \frac{1}{3}$

(d) $\binom{6}{4}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^2$

79. (a) $P\{X = 0\} = \dfrac{\binom{94}{10}}{\binom{100}{10}}$

(b) $P\{X > 2\} = 1 - \dfrac{\binom{94}{10} + \binom{94}{9}\binom{6}{1} + \binom{94}{8}\binom{6}{2}}{\binom{100}{10}}$

Theoretical Exercises

14. (a) $1 - \sum_{n=1}^{\infty} \alpha p^n = 1 - \frac{\alpha p}{1-p}$

(b) Condition on the number of children: For $k > 0$

$$P\{k \text{ boys}\} = \sum_{n=1}^{\infty} P\{k|n \text{ children}\}\alpha p^n$$
$$= \sum_{n=k}^{\infty} \binom{n}{k}(1/2)^n \alpha p^n$$

$$P\{0 \text{ boys}\} = 1 - \frac{\alpha p}{1-p} + \sum_{n=1}^{\infty} \alpha p^n (1/2)^n$$

20. Let S denote the number of heads that occur when all n coins are tossed, and note that S has a distribution that is approximately that of a Poisson random variable with mean λ. Then, because X is distributed as the conditional distribution of S given that $S > 0$,

$$P\{X = 1\} = P\{S = 1 \mid S > 0\} = \frac{P\{S = 1\}}{P\{S > 0\}} \approx \frac{\lambda e^{-\lambda}}{1 - e^{-\lambda}}$$

27. $$P\{X = n + k \mid X > n\} = \frac{P\{X = n + k\}}{P\{X > n\}}$$
$$= \frac{p(1-p)^{n+k-1}}{(1-p)^n}$$
$$= p(1-p)^{k-1}$$

If the first n trials are fall failures, then it is as if we are beginning anew at that time.

32. $$P\{X = k\} = \frac{k-1}{n}\prod_{i=0}^{k-2}\frac{n-i}{n}, \; k > 1$$