1. (a) The joint probability function of X and Y is given as :

P(X=x, Y=y)		Y = 2	Y = 4	Y = 6
X	1	0.05	0.14	0.10
	2	0.25	0.10	0.02
	3	0.15	0.17	0.02

Find P(XY≤6).

(b) X and Y have a correlation coefficient of $\frac{2}{3}$, and Var(X) = 1, Var (Y) = 9. Find Var(3X - 5Y + 7).

(a) $P(XY \leq 6) = P(X=1, Y=2 \cup X=2, Y=2 \cup X=3, Y=2, \cup X=1, Y=4 \cup$
$\cup\ X=1, Y=6) =$

disjoint → = $\cancel{P}$ $P(X=1, Y=2) + P(X=2, Y=2) + P(X=3, Y=2) +$
$P(X=1, Y=4) + P(X=1, Y=6) =$

$= 0.05 + 0.25 + 0.15 + 0.14 + 0.1 = 0.69$

(b) $\rho_{XY} = \frac{2}{3} = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{Cov(X,Y)}{3} \Rightarrow Cov(X,Y) = 2$

$Var(3X-5Y+7) = Var(3X) + Var(5Y) - 2\,Cov(3X, 5Y) =$

$= 9Var(X) + 25Var(Y) - 30\,Cov(X,Y) =$

$= \underbrace{9 \times 1}_{9} + \underbrace{25 \times 9}_{225} - \underbrace{30 \times 2}_{60} = 9 + 225 - 60 = 174$

2. An insurance company provides insurance to three groups of staff with the following characteristics:

	Age	Number of individuals	Probability of making a claim(in a year)	Expected amount per claim (in a year)
1	Below 30	20	0.02	\$500
2	30 to 50	50	0.04	\$1000
3	Above 50	30	0.06	\$1500
		=100		

There are 100 individuals in the portfolio.

(a) What is the probability that a randomly selected individual is below the age of 30 and will make a claim in a year?

(b) What is the probability that a randomly selected individual will not make any claim in a year?

(c) Given that the randomly selected individual has made a claim in a year, what is the probability that he/she is older than 50?

(d) Determine the expected total claim amount of all 100 individuals in a year.

1(a) $X = \{1 \text{ if below } 30, 0 \text{ o/w}\}$, $Y = \{1 \text{ if make a claim}, 0 \text{ o/w}\}$

$$P(X=1, Y=1) = P(Y=1 / X=1) P(X=1) = 0.02 \times 0.2 = 0.004$$

(b) $P(Y=0) = P(Y=0 \mid \text{below } 30) P(\text{Below } 30) + P(Y=0 \mid 30 \text{ to } 50) P(30 \text{ to } 50) +$

$+ P(Y=0 \mid >50) P(>50) = 0.98 \times 0.2 + 0.96 \times 0.5 + 0.94 \times 0.3 =$

$= 0.196 + 0.48 + 0.282 = 0.958$

(c) $X_i = \{1 \text{ if belongs to group } i\}$

$$P(X_3=1 / Y=1) = \frac{P(Y=1 / X_2=1) P(X_3=1)}{\sum_{i=1}^{3} P(Y=1 / X_i=1) P(X_i=1)} =$$

$$= \frac{\cancel{0.06 \times 0.3} \; 0.06 \times 0.3}{0.2 \times 0.02 + 0.04 \times 0.5 + 0.06 \times 0.3} \simeq 0.4286$$

(d) $E(\text{Claim}) = 20 \times 0.02 \times 500 + 50 \times 0.04 \times 1000 + 30 \times 0.06 \times 1500 = \$4900$

3. (a) A fair coin is tossed independently 20 times. Let X be the total number of heads in the first 10 tosses, and let Y be the total number of heads in the complete set of 20 tosses.

   i. Given that Y = 12, what is the (conditional) probability that X = 6?

   ii. Find the expected value of X if Y = 12.

   (b) i. A statistics instructor has put 4 copies of his lecture notes in the library reserve section. Let X represents the number of students who come in to borrow these notes. Suppose X has a Poisson distribution with parameter $\lambda=3$. Find the expected number of copies that have been borrowed. (Note: No more than 4 copies in total can be borrowed).

   ii. The moment generating function of a Poisson random variable X is given as $M_X(t)=e^{(5e^t-5)}$. Calculate $P(X\neq 5)$.

(a) $X \sim Bin(10, 0.5)$ — indep — $Z = Y - X \sim Bin(10, 0.5)$ Also, $X|Y=12 \sim Hyper(20,12,1$

$Y \sim Bin(20, 0.5)$

i. $P(X=6|Y=12) = \dfrac{P(Y=12, X=6)}{P(Y=12)} = \dfrac{P(Z=6, X=6)}{P(Y=12)} =$

$= \dfrac{P(Z=6)\,P(X=6)}{P(Y=12)} = \dfrac{\left[\binom{10}{6} 0.5^{10}\right]^2}{\binom{20}{12} 0.5^{20}} = \dfrac{\binom{10}{6}^2}{\binom{20}{12}} \doteq 0.35$

ii. $E(X|Y=12)$

$X|Y=12 \sim Hypergeom(N=20, M=12, n=10)$

$\Rightarrow E(X|Y=12) = \dfrac{10 \times 12}{20} = 6 \quad = \dfrac{nM}{N}$

(b) i. $X \sim Poisson(3)$, $Y = \{\#\text{ copies borrowed}\}$

$E(Y) = \sum_{i=0}^{3} i\, P(X=i) + 4 \times P(X \geq 4) = 0 + \dfrac{e^{-3}3^1}{1!} + 2\dfrac{e^{-3}3^2}{2!} + 3\dfrac{e^{-3}3^3}{3!} +$

$+ 4\left(1 - \dfrac{e^{-3}3^0}{0!} - \dfrac{e^{-3}3^1}{1!} - \dfrac{e^{-3}3^2}{2!} - \dfrac{e^{-3}3^3}{3!}\right) =$

$\doteq 2.68$

③ (b) ii. $E(X) = M'_X(t)\Big|_{t=0}$

$M'_X(t) = 5\, e^{(5e^t - 5)}\, e^t \Rightarrow M'_X(0) = 5 \Rightarrow \lambda = 5 = E(X)$

$P(X \neq 5) = 1 - \dfrac{e^{-5}\, 5^5}{5!} = 0.8245$

4. (a) Twenty random variables are independently observed from a uniform (0,1) distribution. What is the approximate probability that their sum is at least 8? (Hint: use the CLT).

   (b) Each time Jim buys an item using his credit card, he keeps a record of that expense in his expense account. He always rounds off the amount he enters into his account UPWARDS to the nearest dollar. Thus, if he spends \$3.42, he enters \$4 in his account, etc. At the end of one year, he has used his credit card 200 times. What is the probability that the total in his expense account will be more than \$105 higher than his true total expenses? What assumptions must be made to allow the use of the CLT in this problem?

(a) $X_1, \dots, X_{20} \overset{iid}{\sim} U(0,1) \rightarrow E(X_i) = 0.5, \; Var(X_i) = \frac{1}{12}$

$S = \sum_{i=1}^{20} X_i \overset{\sim}{\approx} N\left(20 \times 0.5, \, 20 \times \frac{1}{12}\right)$ by CLT

$S \overset{\sim}{\approx} N\left(10, \frac{5}{3}\right)$

$P(S \geq 8) \cong P\left(\frac{S-10}{\sqrt{5/3}} \geq \frac{8-10}{\sqrt{5/3}}\right) \simeq P(Z \geq -1.5492) = 0.9393$

(b) Assume all ~~pieces~~ cents are equally likely $\Rightarrow C_i = \{\#$ ~~change~~ amount rounded up in transaction $i\}$

$C_1, \dots, C_{200} \sim U(0,1)$

$S = \sum_{i=1}^{200} C_i \overset{\sim}{\approx} N\left(100, \frac{50}{3}\right)$ by CLT

$P(S > 105) \cong P\left(\frac{S-100}{\sqrt{50/3}} > \frac{105-100}{\sqrt{50/3}}\right) \simeq P(Z > 1.225) = 0.1103$

5. Let X and Y be two independent uniform (0,1) random variables. Find the joint probability density function of U and V, where U = -2 ln X and V = -2 ln Y.

$U = -2 \ln X \Leftrightarrow X = e^{-\frac{U}{2}}$, $Y = e^{-\frac{V}{2}}$

$F_{UV}(u,v) = P(U \le u, V \le v) = P(X \ge e^{-\frac{u}{2}}, Y \ge e^{-\frac{v}{2}}) = P(X \ge e^{-\frac{u}{2}}) P(Y \ge e^{-\frac{v}{2}})$

$= [1 - F_X(e^{-\frac{u}{2}})][1 - F_Y(e^{-\frac{v}{2}})] = (1 - e^{-\frac{u}{2}})(1 - e^{-\frac{v}{2}})$

$f_{UV}(u,v) = \frac{\partial^2 F_{UV}(u,v)}{\partial u \, \partial v} = \frac{\partial}{\partial v}\left[(1 - e^{-\frac{v}{2}})(\frac{1}{2} e^{-\frac{u}{2}})\right] = e^{-\frac{u}{2}} e^{-\frac{v}{2}} \frac{1}{4}$, $\forall u, v > 0$

6. A box contains 15 balls (4 Yellow, 5 Purple and 6 Blue). Suppose 3 balls are drawn at random

   (a) with replacement. Calculate P(all 3 balls are of the same colour).
   (b) without replacement. Calculate P(all 3 balls are of the same colour).
   (c) with replacement. Calculate P(all 3 balls are of different colours).
   (d) without replacement. Calculate P(all 3 balls are of different colours).

(b) $P(\text{all} = \text{colour}) = P(\text{all } Y) + P(\text{all } P) + (\text{all } B) =$

$$= \frac{\binom{4}{3} + \binom{5}{3} + \binom{6}{3}}{\binom{15}{3}} = 0.0747$$

(a) $P(\text{all} = \text{colour}) = P(\text{all } Y) + P(\text{all } P) + P(\text{all } B) = \left(\frac{4}{15}\right)^3 + \left(\frac{5}{15}\right)^3 + \left(\frac{6}{15}\right)^3 =$

$= 0.12$

(c) $P(\text{all} \neq \text{colours}) = \frac{3! \times 4 \times 5 \times 6}{15^3} = 0.2\overline{13}$

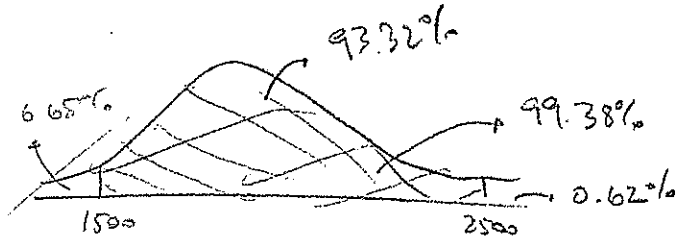
(d) $P(\text{all} \neq \text{colours}) = \frac{\binom{4}{1}\binom{5}{1}\binom{6}{1}}{\binom{15}{3}} = 0.2637$

7. A company manufactures transistors of a certain type that have an average life-span of μ hours and a standard deviation of σ hours. From past experience, we know

   93.32% of these transistors will have a life-span of at least 1500 hours;

   99.38% of these transistors will have a life-span of at most 2500 hours.

   Assume the distribution of life-spans is normal. Estimate the mean μ and the standard deviation σ.



$X \sim N(\mu, \sigma^2)$

↳ life-span

$$P\left(\frac{X-\mu}{\sigma} \leq \frac{1500-\mu}{\sigma}\right) = 0.0668 \qquad P\left(\frac{X-\mu}{\sigma} \geq \frac{2500-\mu}{\sigma}\right) = 0.0062$$

$$\Rightarrow \begin{cases} \dfrac{1500-\mu}{\sigma} = -1.5 \\ \dfrac{2500-\mu}{\sigma} = 2.5 \quad \text{— (1)} \end{cases} \Rightarrow \frac{1500-2500}{\sigma} = -1.5-2.5$$

$$\Rightarrow -1000 = -4\sigma$$

$$\Rightarrow \sigma = 250$$

$\mu = 1875$ from (1)

8. Let X denote the time to failure (in years) of a certain hydraulic component. Suppose the probability density function of X is

$$f(x) = \begin{cases} 0.04x & 0 \le x < 5 \\ 0.4 - 0.04x & 5 \le x < 10 \\ 0 & \textit{otherwise} \end{cases}$$

(a) Find the probability that time to failure is between 3 and 7 years.

(b) Find the expected failure time.

(c) Find the median failure time.

a) $P(3 < x < 7)$

$= P(3 < x < 5) + P(5 \le x < 7)$

$= \int_3^5 0{\cdot}04\, x\, dx + \int_5^7 ({\cdot}4 - 0{\cdot}04x)\,dx$

$= 0{\cdot}04 \frac{x^2}{2} \Big|_3^5 + 0{\cdot}4\, x \Big|_5^7 - 0{\cdot}04 \frac{x^2}{2} \Big|_5^7$

$= {\cdot}32 + {\cdot}8 - {\cdot}48 = {\cdot}64$

b) $E(X) = \int_3^5 x\, 0{\cdot}04\, x\, dx + \int_5^7 x\,({\cdot}4 - 0{\cdot}04x)\,dx$

$= \frac{x^3}{3} 0{\cdot}04 \Big|_3^5 + 0{\cdot}4 \frac{x^2}{2} \Big|_5^7 - 0{\cdot}04 \frac{x^3}{3} \Big|_5^7 = {\cdot}3$

c) $P(X \le m) = 0{\cdot}5 = \int_{-\infty}^{x} f(x)\,dx$

median

$\int_0^5 0{\cdot}04\, x\, dx = 0{\cdot}04 \frac{x^2}{2} \Big|_0^5 = 0{\cdot}04 \frac{25}{2} = {\cdot}5$

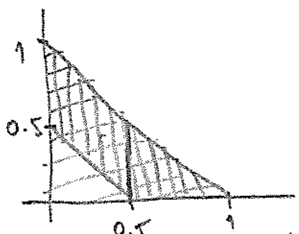
Hence 5 is the median, m

9. (a) The joint probability density function of X and Y is given by f(x,y) = 3(x+y), for 0<x+y<1, 0<x<1, 0<y<1, and f(x,y) = 0 otherwise.
   i. Find the marginal probability density of X.
   ii. Find P[X + Y > 0.5].

   (b) X and Y are both continuous random variables. Given Y=y, X is uniformly distributed in the interval from 0 to y. If $f_Y(y)=2y$ for $0< y <1$, find E(X).

i. $f_X(x) = \int_0^{1-x} 3(x+y)\,dy = 3\left[xy\Big|_0^{1-x} + \frac{y^2}{2}\Big|_0^{1-x}\right] = 3x(1-x) + \frac{3}{2}(1-x)^2$

ii.



$$P(X+Y>0.5) = \int_{0.5}^{1}\int_0^{1-x} 3(x+y)\,dy\,dx + \int_0^{0.5}\int_{0.5-x}^{1-x} 3(x+y)\,dy\,dx =$$

$$= 3\left[\int_{0.5}^{1} xy + \frac{y^2}{2}\Big|_0^{1-x} dx + \int_0^{0.5} xy + \frac{y^2}{2}\Big|_{0.5-x}^{1-x} dx\right] =$$

$$= 3\left[\int_{0.5}^{1} (1-x)x + \frac{(1-x)^2}{2}\,dx + \int_0^{0.5} x(1-x) + \frac{(1-x)^2}{2} - x\left(\frac{1}{2}-x\right) - \frac{\left(\frac{1}{2}-x\right)^2}{2}\,dx\right] =$$

$$= 3\left[\frac{1}{2}\left(\int_0^1 1-x^2\,dx - \int_0^{0.5} \frac{1}{4}-x^2\,dx\right)\right] = \frac{3}{2}\left[\left(x-\frac{x^3}{3}\right)\Big|_0^1 - \left(\frac{x}{4}-\frac{x^3}{3}\right)\Big|_0^{0.5}\right] =$$

$$= \frac{3}{2}\left[\frac{2}{3} - \frac{1}{8} + \frac{1}{24}\right] = \frac{3}{2}\left(\frac{16-3+1}{24}\right) = \frac{\cancel{3}}{\cancel{2}}\cdot\frac{\cancel{14}^{7}}{\cancel{24}_{8}} = \frac{7}{8}$$

9. (a) The joint probability density function of X and Y is given by f(x,y) = 3(x+y), for $0<x+y<1$, $0<x<1$, $0<y<1$, and f(x,y) = 0 otherwise.

   i. Find the marginal probability density of X.

   ii. Find P[X + Y > 0.5].

   (b) X and Y are both continuous random variables. Given Y=y, X is uniformly distributed in the interval from 0 to y. If $f_Y(y)=2y$ for $0< y <1$, find E(X).

b)
$X \mid Y=y \sim \text{Unif}(0, y)$

$f_Y(y) = 2y$

$E(X) = E(E(X \mid Y))$

$= E\left(\frac{y-0}{2}\right) = E\left(\frac{y}{2}\right) = \frac{1}{2} E(y)$

$E(y) = \int_0^1 y \cdot 2y \, dy = \int_0^1 2y^2 \, dy = 2 \left. \frac{y^3}{3} \right|_0^1$

$= \frac{2}{3}$

$\therefore E(X) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$