THE UNIVERSITY OF BRITISH COLUMBIA
FINAL EXAMINATION
April 1999

MATH / STAT 302
Introduction to Probability

TIME: 2 ½hours

Special Instructions: You are allowed to use a calculator and one page of notes. There are 10 problems for a total of 100 points. For questions 4 to 7, choose one of part I or II. All questions are worth 10 points each. All answers must appear in the answer blanks where provided. Work must be shown in the spaces provided. Be brief but mathematically precise. Provide just enough detail to make your answer clear to the examiners.

Question #	1	2	3	4	5	6	7	8	9	10
Mark										

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1. A certain firm produces resistors and markets them as 10-ohm resistors. However the actual ohms of resistance by the resistors may vary. Research has established that 10% of the values are below 9.5 ohms and 20% are above 10.5 ohms. If two resistors, randomly selected, are used in a system, find the probabilities of the following events:

   (a) Both resistors have actual values between 9.5 and 10.5 ohms.

   (b) At least one resistor has an actual value in excess of 10.5 ohms.

2. Two assembly lines (I and II) have the same rate of defectives in their production of voltage regulators. Five regulators are sampled from each line and tested. Among the total of 10 tested regulators, 4 are defective. Find the probability that exactly 3 of the defectives came from line I.

3. Two sentries are sent to patrol a road that is 1 km long. The sentries are sent to points chosen independently and at random along the road. Find the probability that the sentries will be less than ¼ km apart when they reach their assigned posts.

4. **PART I.** A large construction firm has won 60% of the jobs for which it has bid. Suppose this firm bids on 25 jobs next month. Approximate the probability that it will win at least 20 of these jobs.

   PART II. For a certain section of pine forest, the number Y of diseased trees per acre has a Poisson distribution with mean $\lambda = 10$. The diseased trees are sprayed with an insecticide at a cost of \$3 per tree, plus a fixed overhead cost for equipment rental of

\$50. Letting C denote the total spraying cost for a randomly selected acre, find the expected value and standard deviation for C.

5. **PART I.** A dart lands at random coordinates (X,Y) on the $[0.1] \times [0.1]$-square. Find the density of the area enclosed between the points (0,0) and (X, Y).

   PART II. A dart lands on a point (X, Y) that is uniformly distributed over the surface of a dart board with unit radius and "bulls-eye" placed at the origin. [In other words the dart board is the set of all (x,y) for which $x^2 + y^2 \leq 1$.] Let $R = \sqrt{X^2 + Y^2}$ denote the distance of the dart from the bull's eye (*i.e.* origin).

   a) Find the density of function of R.

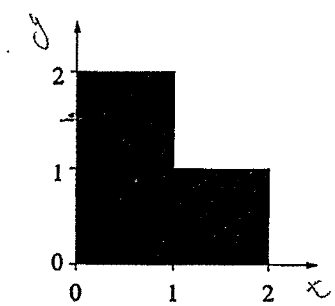
   b) Find $E(R)$.

6. **PART I.** An electronic surveillance system has one of each of two different types of components in joint operations. If X and Y denote the random life lengths of the components of type I and type II, respectively, the joint probability density function is given by $f(x,y) = k\,x \exp\{-(x + y)/2\}$ if $x > 0$ and $y > 0$ and it is 0 otherwise where k = 1/8. TEXT 5.11

   (a) Are X and Y independent? [State why or why not.]

   (b) Find $P(X > 1, Y > 1)$.

   PART II. Consider the joint density $f_{x,y}$ of X and Y which is constant on the shaded area in the following figure

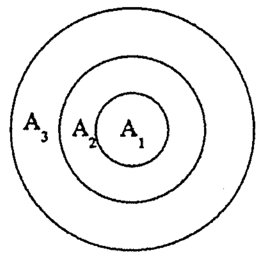


   (a) Find the distribution of X given $Y = y$.

   (b) Compute $E(X \mid Y = 3/2) =$ _______ and Var(X/Y = 3/2)= _______

   (c) Suppose you'll have to guess the value of X after being told the value of Y. You will win a larger amount if your guess is close to the truth. Would you rather face the situation where you are told (choose one) _$Y = ½$ or _$Y = 3/2$? Explain.

7. **PART I.** Five darts are thrown on the target below and land in one of the areas A_1, A_2 or A_3 with probabilities $p_1 = 1/6$, $p_2 = 1/3$, $p_3 = 1/2$ respectively. Find the probability that three of the five darts land in area A_1 or A_2.



PART II. Three power board assemblers B_1, B_2, and B_3 produce power boards at varying rates of speed and at varying errors rates. For example, 2% of B_1's boards are defective.
The following table gives a complete summary.

Assembler	B_1	B_2	B_3
% of boards s/he makes	25%	45%	30%
% of his/her boards that are defective	2%	3%	2%

A board is randomly drawn off the assembly line and found to be defective. What is the probability it was made by B_2.

8. A system with n parallel components will continue to function during the next 400 days as long as at least 1 component functions. The components operate independently and each will survive the next 400 days with probability exp (-1/2000).

   (a) How big must n be to insure a probability of at least 1 – exp (-1/10) that the system survives the next 400 days.

   (b) Repeat (a) using a convenient and valid approximation.

9. Suppose X and Y have the joint distribution given by $f(x,y) = (x + y)$ for $0 \leq x, y \leq 1$ and 0 otherwise. Find the distribution of X+Y.

10. (a) A random variable X is uniformly distributed over the unit interval (0,1). Find the distribution of $-\ell n X$, ℓn noting the natural logarithm.

    (b) Suppose $X_1, \ldots, X_n$ are n such uniform variables (all independent). State what happens in precise mathematical terms to the quantity $G = (X_1 \times \ldots \times X_n)^{1/n}$ as n tends to ∞.

1. $X_i = \{\text{ohms of resistance of resistor } i\}$, $P(X_i<9.5)=0.1$, $P(X_i>10.5)=0.2$

(a) $P(9.5<X_1<10.5, 9.5<X_2<10.5) \overset{\text{ind}}{=} [P(9.5<X_i<10.5)]^2 =$

$= [1-0.1-0.2]^2 = 0.7^2 = 0.49$

(b) $P(X_1>10.5 \cup X_2>10.5) = P(X_1>10.5)+P(X_2>10.5) - P(X_1>10.5, X_2>10.5) =$

$\overset{\text{ind}}{=} 0.2+0.2-[0.2]^2 = 0.36$

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2. $X = \#$ def from line I $\sim Bin(n=5, p)$
$Y =$ " " " II $\sim Bin(n=5, p)$

$W = X+Y$ = total # def. $\sim Bin(n=10, p)$

$P(X=3 \mid W=4)$

$$= \frac{P(X=3 \cap W=4)}{P(W=4)}$$

$$= \frac{P(W=4 \mid X=3)\,P(X=3)}{P(W=4)}$$

$$= \frac{P(Y=1)\,P(X=3)}{P(W=4)}$$

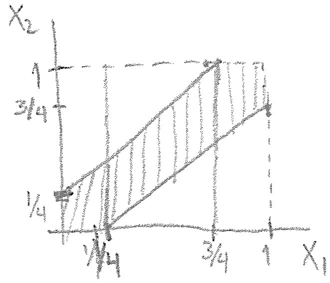
$$= \frac{\binom{5}{1} p^1 \cancel{(1-p)^4} \binom{5}{3} p^3 \cancel{(1-p)^2}}{\binom{10}{4} p^4 \cancel{(1-p)^6}} = \frac{\binom{5}{1}\binom{5}{3}}{\binom{10}{4}} = \frac{5 \times 10}{210}$$

$= 0.2381$

(3) $X_1 = \{\text{location of } 1^{st} \text{ sentry}\}$

$X_2 = \{\text{location of } 2^{nd} \text{ sentry}\}$

$\Rightarrow X_1, X_2 \overset{iid}{\sim} U(0,1)$



We want $P(|X_1 - X_2| < \frac{1}{4}) \to$ shaded area

$f_{X_1X_2}(x_1, x_2) = 1$ for $(x_1, x_2) \in (0,1)^2$ since X_1, X_2 are independent uniforms

$$P(|X_1 - X_2| < \tfrac{1}{4}) = \int_0^{1/4}\int_0^{x_1+\frac{1}{4}} 1\, dx_2\, dx_1 + \int_{1/4}^{3/4}\int_{x_1-\frac{1}{4}}^{x_1+\frac{1}{4}} 1\, dx_2\, dx_1 + \int_{3/4}^{1}\int_{x_1-\frac{1}{4}}^{1} 1\, dx_2\, dx_1 =$$

$$= \int_0^{1/4} x_1 + \frac{1}{4}\, dx_1 + \int_{1/4}^{3/4} \frac{1}{2}\, dx_1 + \int_{3/4}^{1} 1 - \left(x_1 - \frac{1}{4}\right) dx_1 =$$

$$= \frac{x_1^2}{2} + \frac{x_1}{4}\Bigg|_0^{1/4} + \frac{x_1}{2}\Bigg|_{1/4}^{3/4} + \frac{5x_1}{4} - \frac{x_1^2}{2}\Bigg|_{3/4}^{1} =$$

$$= \frac{1}{32} + \frac{1}{16} + \frac{3}{8} - \frac{1}{8} + \frac{5}{4} - \frac{15}{16} - \frac{1}{2} + \frac{9}{32} = \frac{1+2+12-4+40-30-16+9}{32} =$$

$$= \boxed{\frac{7}{16}}$$

(4) I) X = {# of jobs won out of 25 bids next month} , $X \sim Bin(25, 0.6)$

Normal App: $X \approx N(15, 15 \times (0.4))$, $X \approx N(15, 6)$ $\quad np = 15$, $(1-p)n = 10 \geq 5$

$P(X \geq 20) \approx P\left(Z > \frac{19.5 - 15}{\sqrt{6}}\right) = P(Z > 1.8371) = 0.0331$

(4) II) $Y \sim Poisson(10)$

$C = 3Y + 50$

$E(C) = E(3Y + 50) = 3E(Y) + 50 = 3 \times 10 + 50 = 80$

$Var(C) = Var(3Y + 50) = 3^2 Var(Y) = 9 \times 10 = 90 \Rightarrow SD(C) = \sqrt{90} \approx 9.49$

Part I:

6. a) $f(x,y) = \frac{1}{8} x e^{-\frac{x+y}{2}}$

$$= \underbrace{\frac{1}{8} x e^{-\frac{x}{2}}}_{h(x)} \underbrace{e^{-\frac{y}{2}}}_{g(y)}$$

As it is possible to factor them as $f(x,y) = h(x)g(y)$, we can say they are independent.

b) $P(X>1, Y>1) = \int_1^\infty \int_1^\infty \frac{1}{8} x e^{-\frac{x+y}{2}} \, dx dy$

$x>0, y>0$

solvable by integration by parts.

⑥ Part II) Since Area $= 3 \Rightarrow f_{XY}(x,y) = \frac{1}{3}$ for $(x,y) \in$ Shaded Area

(a) $X|Y=y \sim U(0,1)$ if $1<y<2$

$X|Y=y \sim U(0,2)$ if $0<y\leq 1$

$$\Rightarrow \quad f_{X|Y=y}(x) = \begin{cases} 1/2 & \text{if } 0<x<2 \text{ for } y\in(0,1) \\ 1 & \text{if } 0<x<1 \text{ for } y \in (1,2) \end{cases}$$

(b) $X|Y=\frac{3}{2} \sim U(0,1) \Rightarrow E(X|Y=\frac{3}{2}) = 0.5$

$\mathrm{Var}(X|Y=\frac{3}{2}) = \frac{1}{12}$

(c) $Y=\frac{3}{2}$! Than ~~X~~ is between 0 & 1, while X is in $(0,2)$ for $Y=\frac{1}{2}$

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⑦ I) $P(1 \text{ dart in area } A_1 \text{ or } A_2) = P_1 + P_2 = \frac{1}{2}$

$X = \{\#\text{ darts in areas } A_1 \text{ or } A_2\} \Rightarrow X \sim \mathrm{Bin}(5, \frac{1}{2})$

$P(X=3) = \binom{5}{3} 0.5^5 = 0.3125$

II) $$P(B_2 \mid \text{defective}) = \frac{P(\text{defective} \mid B_2)\, P(B_2)}{\sum_{i=1}^{3} P(\text{defective} \mid B_i)\, P(B_i)} = \frac{0.03 \times 0.45}{0.03\times 0.45 + 0.02\times 0.25 + 0.02\times 0.3}$$

$= 0.551$

8. $X_i = \{\text{life time of component } i\}$

$P(X_i \geq 400) = e^{-\frac{1}{2000}}$

$S = \{\text{life time of system}\}$

$= \max\{X_1, X_2 \ldots X_n\}$

a) $P(S \geq 400) = P\{\max(X_1 \ldots X_n) \geq 400\}$

$= P\left(\bigcup_{i=1}^{n} X_i \geq 400\right) = 1 - P\left(\bigcap_{i=1}^{n} X_i < 400\right)$

$\overset{iid}{=} 1 - P(X_1 < 400)\, P(X_2 < 400) \ldots P(X_n < 4$

$= 1 - P(X_i < 400)^n = 1 - \left(1 - e^{-\frac{1}{2000}}\right)$

$\therefore\ P(S \geq 400) \geq 1 - e^{-\frac{1}{10}}$

$\Rightarrow 1 - \left(1 - e^{-\frac{1}{2000}}\right)^n \geq 1 - e^{-\frac{1}{10}}$

$\Rightarrow \left(1 - e^{-\frac{1}{2000}}\right)^n \leq e^{-\frac{1}{10}}$

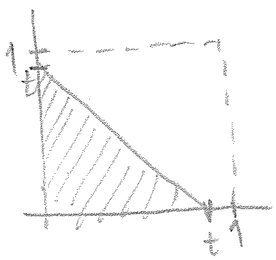
$\Rightarrow n \log\left(1 - e^{-\frac{1}{2000}}\right) \leq -\frac{1}{10}$

$\Rightarrow n \geq -\frac{1}{10} \times \frac{1}{\log\left(1 - e^{-\frac{1}{2000}}\right)} = 0.0131$

$\Rightarrow \boxed{n \geq 1}$

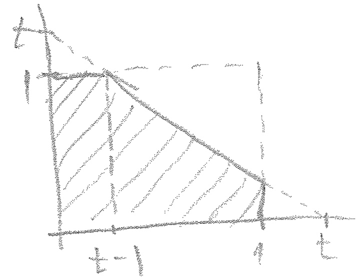
9. $f(x,y) = x+y$, $0 \le x,y \le 1$, Let $T = X+Y$

$\Rightarrow T \in [0,2]$



$t \le 1$

or



$1 < t \le 2$

Then, for $t \le 1$:

$$F_T(t) = P(X+Y \le t) = \int_0^1 \int_0^{t-\cancel{x}} x+y \, dy \, d\cancel{y} = \int_0^1 xy + \frac{y^2}{2} \Big|_0^{t-x} dx = \int_0^1 \frac{(t-\cancel{x})^2}{2} + (t-x)x \, d$$

$$= \frac{1}{2} \int_0^1 t^2 - \cancel{x}^2 \, dx = \frac{1}{2}\left(t^2 x - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{2}\left(t^2 - \frac{1}{3}\right) = \frac{t^2}{2} - \frac{1}{6}$$

for $1 < t \le 2$:

$$F_T(t) = P(X+Y \le t) = \overbrace{\int_0^{t-1} \int_0^1 x+y \, dy \, dx}^{(a)} + \overbrace{\int_{t-1}^1 \int_0^{t-x} x+y \, dy \, dx}^{(b)} =$$

$$(a) = \int_0^{t-1} xy + \frac{y^2}{2} \Big|_0^1 dx = \int_0^{t-1} x + \frac{1}{2} \, dx = \frac{x^2}{2} + \frac{x}{2} \Big|_0^{t-1} = \frac{(t-1)^2}{2} + \frac{t-1}{2} =$$

$$= \frac{1}{2}\left(t^2 - 2t + \cancel{1} + t - \cancel{1}\right) = \frac{1}{2}\left(t^2 - t\right)$$

$$(b) = \int_{t-1}^1 \int_0^{t-x} x+y \, dy \, dx = \int_{t-1}^1 xy + \frac{y^2}{2} \Big|_0^{t-x} dx = \int_{t-1}^1 x(t-x) + \frac{(t-x)^2}{2} \, dx =$$

$$= \int_{t-1}^1 tx - x^2 + \frac{t^2}{2} - tx + \frac{x^2}{2} \, dx = \frac{1}{2} \int_{t-1}^1 t^2 - x^2 \, dx = \frac{1}{2}\left(t^2 x - \frac{x^3}{3} \Big|_{t-1}^1\right) =$$

$$= \frac{1}{2}\left[\left(t^2 - \frac{1}{3}\right) - \left(t^3 - t^2 - \frac{(t-1)^3}{3}\right)\right] = \frac{1}{2}\left[t^2 - \frac{1}{3} - t^3 + \cancel{t^2} + \frac{t^3}{3} - \cancel{t^2} + t - \frac{1}{3}\right] =$$

$$= \frac{1}{2}\left(-\frac{2}{3} t^3 + t^2 + t - \frac{2}{3}\right)$$

Then, $F_T(t) = (a) + (b) = \frac{1}{2}\left[t^2 \cancel{+t} - \frac{2}{3}t^3 + t^2 \cancel{-t} - \frac{2}{3}\right] = \frac{1}{2}\left[-\frac{2}{3}t^3 + 2t^2 - \frac{2}{3}\right] =$

$= t^2 - \frac{t^3}{3} - \frac{1}{3}$

Hence,

$$F_T(t) = P(X+Y \leq t) = \begin{cases} \frac{t^2}{2} - \frac{1}{6} & \text{for } 0 \leq t \leq 1 \\ t^2 - \frac{t^3}{3} - \frac{1}{3} & \text{for } 1 < t \leq 2 \end{cases}$$

Finally, $f_T(t) = F_T'(t)$

So,

$$f_T(t) = \begin{cases} t & \text{for } 0 \leq t \leq 1 \\ 2t - t^2 & \text{for } 1 \leq t \leq 2 \end{cases}$$

(Note that $\lim_{t \to 1^-} f_T(t) = \lim_{t \to 1^+} f_T(t) = 1$)

10 a) let $Y = -\ln X$

$$G_Y(y) = P(Y \leq y) = P(-\ln X \leq y)$$

$$= P(\ln X \geq -y) = P(X \geq e^{-y})$$

$$= 1 - P(X \leq e^{-y}) = 1 - \int_0^{e^{-y}} f(x)\,dx$$

$$= 1 - \int_0^{e^{-y}} 1\,dx = 1 - e^{-y}$$

$$g_y(y) = \frac{d}{dy} G(y) = e^{-y}, \quad 0 < y < \infty$$

b) $\log G = \frac{1}{n} \sum \log X_i = -\frac{1}{n} \sum Y_i$

(when $Y_i = -\log X_i$)

As $n \to \infty$, $\log G \xrightarrow[\text{WLLN}]{\text{weak law of large numbers}} -$ [average of Y_i]

$$\to -E(Y_i)$$

$$= -\int_0^\infty y\, g(y)\, dy$$

$$= -\int_0^\infty y^{2-1} e^{-y}\, dy \to \text{Gamma function}$$

$$= -\Gamma 2 = -(2-1)! = -1$$

$\therefore G \longrightarrow \exp(-E(y_i)) = \exp(-1) = .3678$