Convex Optimization

Machine Learning Summer School

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February 2015

Context: Big Data and Big Models

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  - Seen across many fields of science and engineering.
  - Not gigabytes, but terabytes or petabytes (and beyond).

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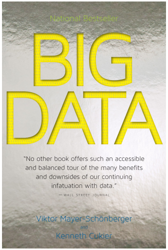

- Many important aspects to the 'big data' puzzle:
  - Distributed data storage and management, parallel computation, software paradigms, data mining, machine learning, privacy and security issues, reacting to other agents, power management, summarization and visualization.

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- Machine learning uses big data to fit richer statistical models:
  - Vision, bioinformatics, speech, natural language, web, social.
  - Developping broadly applicable tools.
  - Output of models can be used for further analysis.

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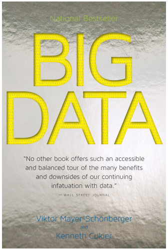

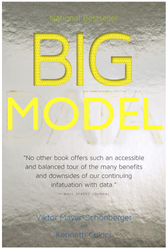

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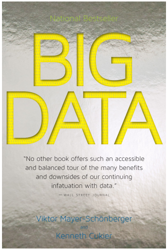

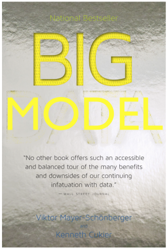

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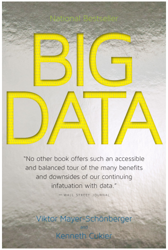

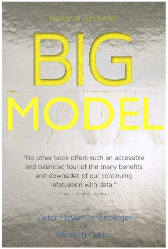

- Numerical optimization is at the core of many of these models.
- But, traditional 'black-box' methods have difficulty with:
  - the large data sizes.
  - the large model complexities.

Motivation: Why Learn about Convex Optimization?

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Why in particular learn about convex optimization?

- Among only *efficiently-solvable* continuous problems.
- You can do a lot with convex models.

  (least squares, lasso, generlized linear models, SVMs, CRFs)
- Empirically effective non-convex methods are often based methods with good properties for convex objectives.

  (functions are locally convex around minimizers)

Two Components of My Research

- The first component of my research focuses on computation:
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  - Can lead to enormous speedups for big data and complex models.

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- We can alternate between these two.

Outline

Convexity: Zero-order condition

A real-valued function is convex if

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y),$$

for all $x, y \in \mathbb{R}^n$ *and all* $0 \leq \theta \leq 1$.

- Function is *below a linear interpolation* from x to y.
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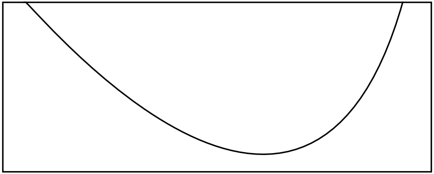

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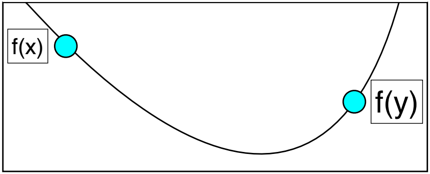

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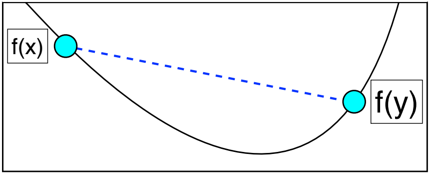

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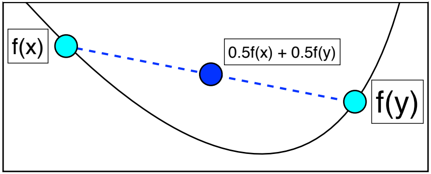

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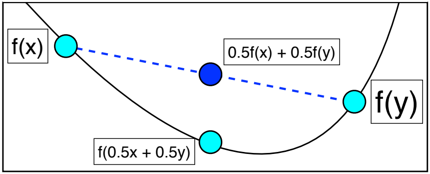

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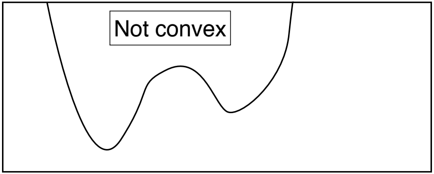

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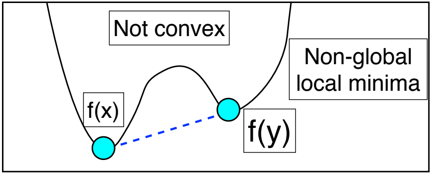

Convexity of Norms

We say that a function f is a **norm** *if:*

1. $f(0) = 0$.
2. $f(\theta x) = |\theta| f(x)$.
3. $f(x + y) \leq f(x) + f(y)$.

Examples:

$$\|x\|_2 = \sqrt{\sum_i x_i^2} = \sqrt{x^T x}$$

$$\|x\|_1 = \sum_i |x_i|$$

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Norms are convex:

$$f(\theta x + (1-\theta)y) \leq f(\theta x) + f((1-\theta)y) \qquad (3)$$

$$= \theta f(x) + (1-\theta) f(y) \qquad (2)$$

Strict Convexity

A real-valued function is strictly convex if

$$f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta)f(y),$$

for all $x \neq y \in \mathbb{R}^n$ *and all* $0 < \theta < 1$.

- *Strictly below the linear interpolation* from x to y.

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- *Strictly below the linear interpolation* from x to y.
- Implies at most one global minimum.

(otherwise, could construct lower global minimum)

Convexity: First-order condition

A real-valued differentiable function is convex iff

$$f(y) \geq f(x) + \nabla f(x)^T (y - x),$$

for all $x, y \in \mathbb{R}^n$.

- The function is globally *above the tangent* at x.

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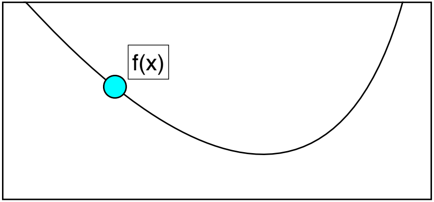

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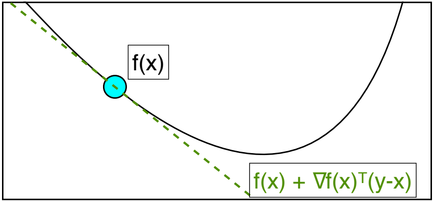

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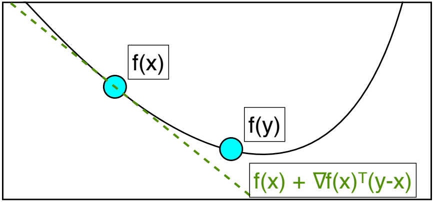

Convexity: Second-order condition

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for all $x \in \mathbb{R}^n$*.*

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A real-valued function f is a quadratic if it can be written in the form:

$$f(x) = \frac{1}{2}x^T Ax + b^T x + c.$$

Since $\nabla^2 f(x) = A$, it is convex if $A \succeq 0$.
E.g., least squares has $\nabla^2 f(x) = A^T A \succeq 0$.

Examples of Convex Functions

Some simple convex functions:

- $f(x) = c$
- $f(x) = a^T x$
- $f(x) = ax^2 + b$ (for $a > 0$)
- $f(x) = \exp(ax)$
- $f(x) = x \log x$ (for $x > 0$)
- $f(x) = ||x||^2$
- $f(x) = \max_i \{x_i\}$

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Some other notable examples:

- $f(x, y) = \log(e^x + e^y)$
- $f(X) = \log \det X$ (for X positive-definite).
- $f(x, Y) = x^T Y^{-1} x$ (for Y positive-definite)

Operations that Preserve Convexity

1. Non-negative weighted sum:

$$f(x) = \theta_1 f_1(x) + \theta_2 f_2(x).$$

2. Composition with affine mapping:

$$g(x) = f(Ax + b).$$

3. Pointwise maximum:

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We know that $\|\cdot\|_p$ is a norm, so it follows from (2).

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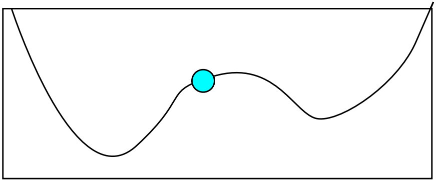

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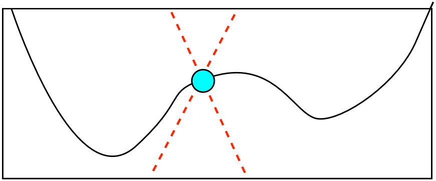

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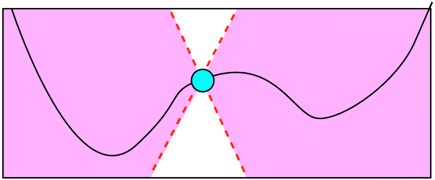

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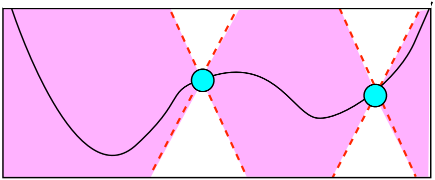

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- Optimization is hard, but assumptions make a big difference.

(we went from impossible to very slow)

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- However, these solvers require $O(n^2)$ or worse cost per iteration.
  - Infeasible for applications where n may be in the billions.
- Solving big problems has led to re-newed interest in simple first-order methods (gradient methods):

$$x^+ = x - \alpha \nabla f(x).$$

  - These only have $O(n)$ iteration costs.
  - But we must analyze how many iterations are needed.

ℓ_2-Regularized Logistic Regression

- Consider ℓ_2-regularized logistic regression:

$$f(x) = \sum_{i=1}^{n} \log(1 + exp(-b_i(x^T a_i))) + \frac{\lambda}{2}\|x\|^2.$$

- Objective f is convex.
- First term is Lipschitz continuous.
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- Second term is not Lipschitz continuous.
- But we have

$$\mu I \preceq \nabla^2 f(x) \preceq LI.$$

$(L = \frac{1}{4}\|A\|_2^2 + \lambda,\ \mu = \lambda)$

- Gradient is Lipschitz-continuous.
- Function is strongly-convex.

(implies strict convexity, and existence of unique solution)

Properties of Lipschitz-Continuous Gradient

- From Taylor's theorem, for some z we have:

$$f(y) = f(x) + \nabla f(x)^T(y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(z)(y - x)$$

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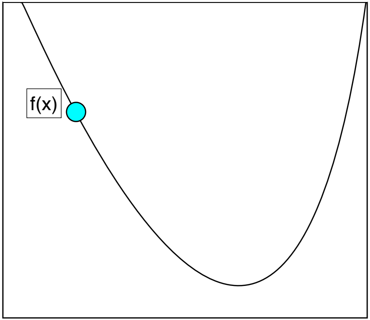

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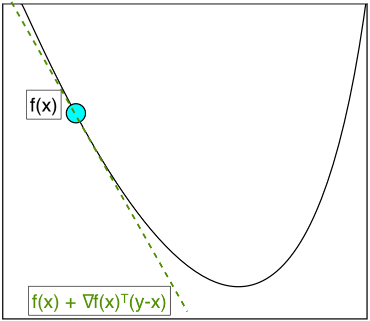

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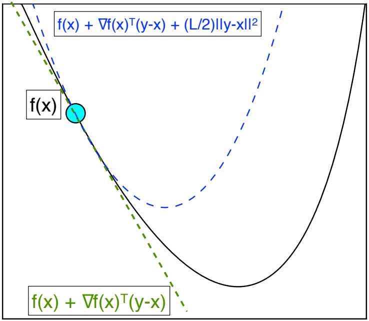

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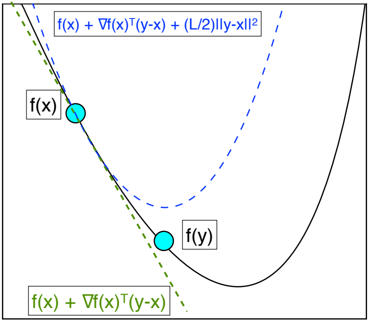

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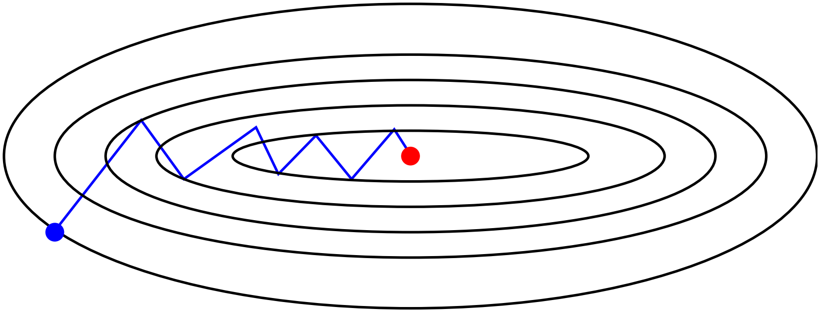

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- Set x^+ to minimize upper bound in terms of y:

$$x^+ = x - \frac{1}{L}\nabla f(x).$$

(gradient descent with step-size of $1/L$)

- Plugging this value in:

$$f(x^+) \leq f(x) - \frac{1}{2L}\|\nabla f(x)\|^2.$$

(decrease of *at least* $\frac{1}{2L}\|\nabla f(x)\|^2$)

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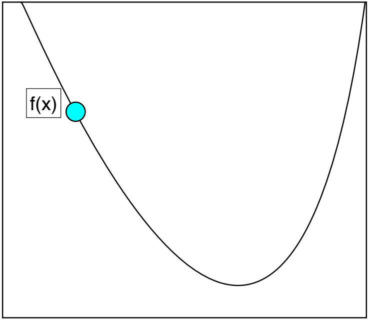

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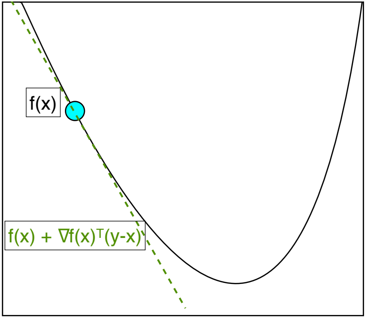

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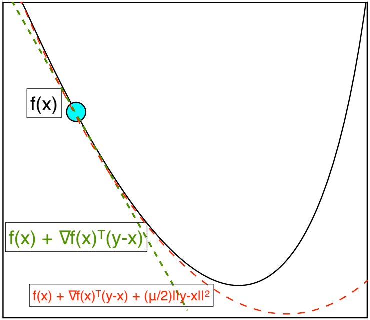

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- Minimize both sides in terms of y:

$$f(x^*) \geq f(x) - \frac{1}{2\mu}\|\nabla f(x)\|^2.$$

- Upper bound on how far we are from the solution.

Linear Convergence of Gradient Descent

- We have bounds on x^+ and x^*:

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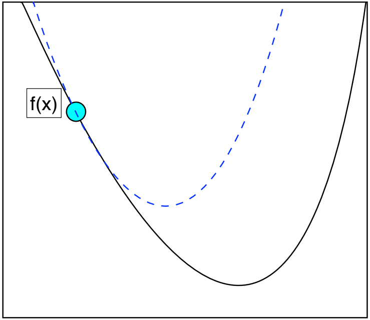

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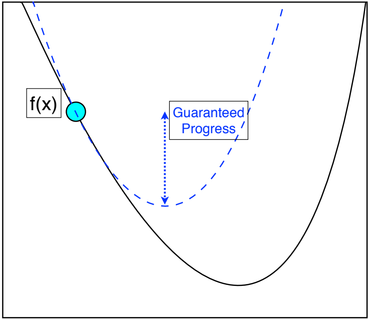

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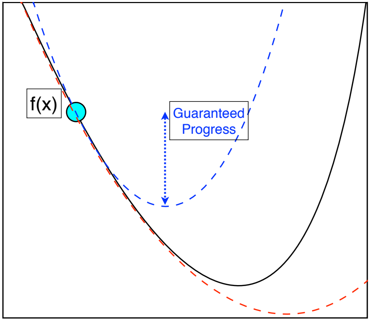

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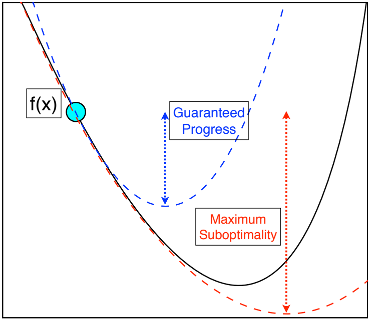

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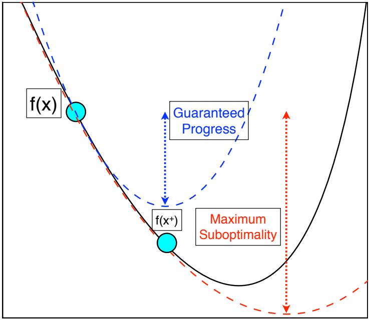

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- This gives a linear convergence rate:

$$f(x^t) - f(x^*) \leq \left(1 - \frac{\mu}{L}\right)^t [f(x^0) - f(x^*)]$$

- Each iteration multiplies the error by a fixed amount.

(very fast if μ/L is not too close to one)

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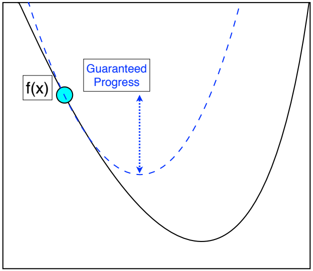

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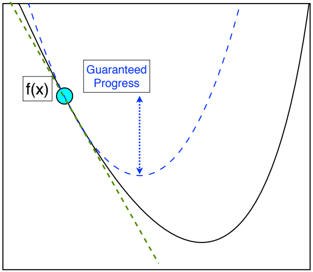

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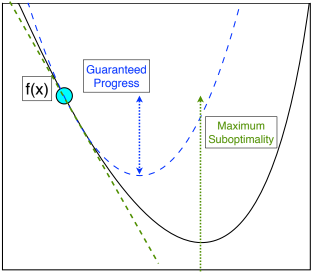

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- If f is convex, then $f + \lambda\|x\|^2$ is strongly-convex.

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- Also, check your derivative code!

$$\nabla_i f(x) \approx \frac{f(x + \delta e_i) - f(x)}{\delta}$$

- For large-scale problems you can check a random direction d:

$$\nabla f(x)^T d \approx \frac{f(x + \delta d) - f(x)}{\delta}$$

Convergence Rate of Gradient Method

We are going to explore the 'convex optimization zoo':

- Gradient method for smooth/convex: $O(1/t)$.
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- 'Error bounds' exist that give linear convergence without strong-convexity. [Luo & Tseng, 1993].
- Is this the best algorithm under these assumptions?

Accelerated Gradient Method

- Nesterov's accelerated gradient method:

$$
\begin{aligned}
x_{t+1} &= y_t - \alpha_t \nabla f(y_t), \\
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  for appropriate α_t, β_t.

- Motivation: "to make the math work"

  (but similar to heavy-ball/momentum and conjugate gradient method)

Convex Optimization Zoo

Algorithm	Assumptions	Rate
Gradient	Convex	$O(1/t)$
Nesterov	Convex	$O(1/t^2)$
Gradient	Strongly-Convex	$O((1-\mu/L)^t)$
Nesterov	Strongly-Convex	$O((1-\sqrt{\mu/L})^t)$

- $O(1/t^2)$ is optimal given only these assumptions.

  (sometimes called the *optimal* gradient method)
- The faster linear convergence rate is close to optimal.
- Also faster in practice, but implementation details matter.

Newton's Method

- The oldest differentiable optimization method is Newton's.

  (also called IRLS for functions of the form $f(Ax)$)

- Modern form uses the update

$$x^+ = x - \alpha d,$$

where d is a solution to the system

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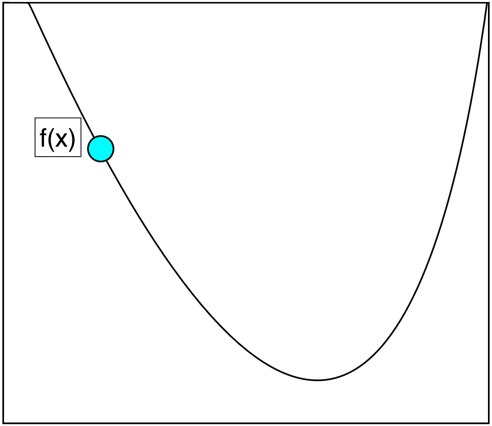
- We can generalize the Armijo condition to

$$f(x^+) \leq f(x) + \gamma\alpha\nabla f(x)^T d.$$

- Has a natural step length of $\alpha = 1$.

  (always accepted when close to a minimizer)

Newton's Method



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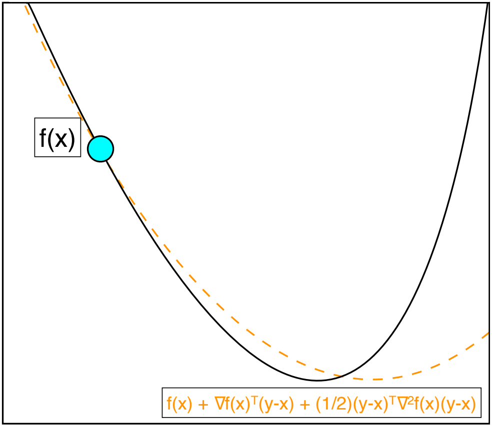

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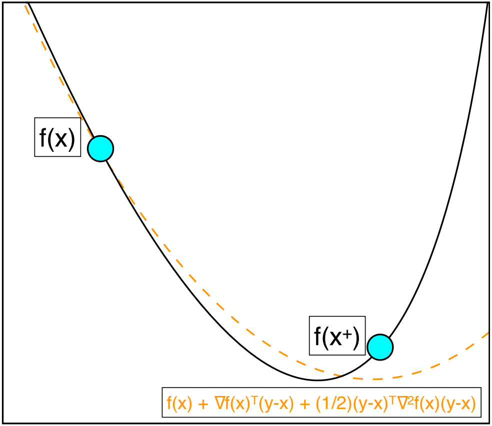

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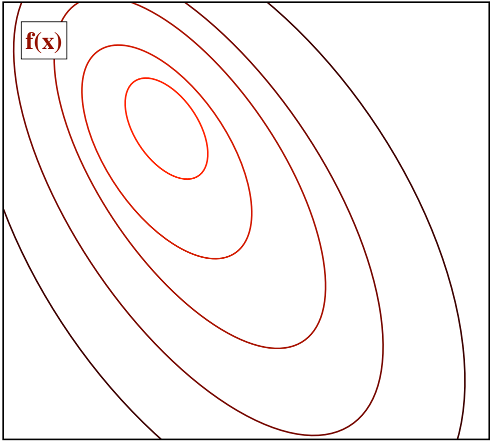

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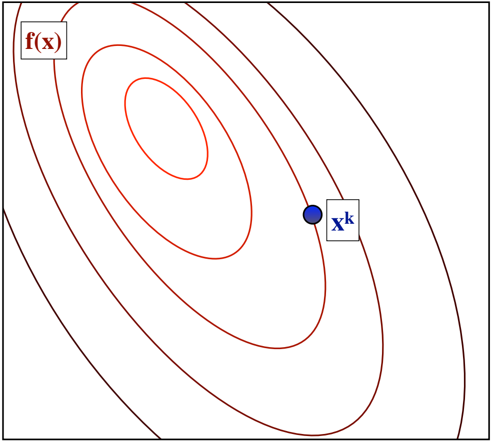

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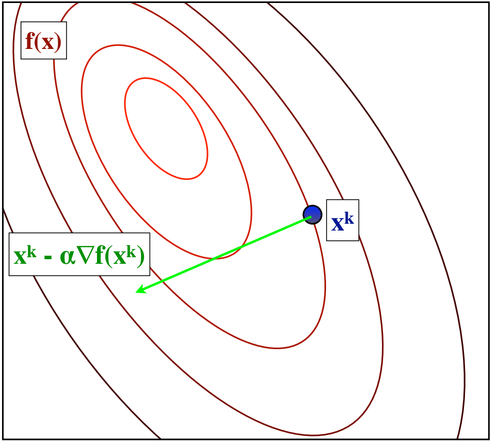

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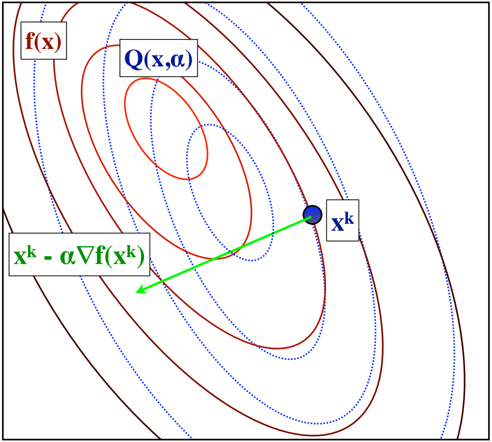

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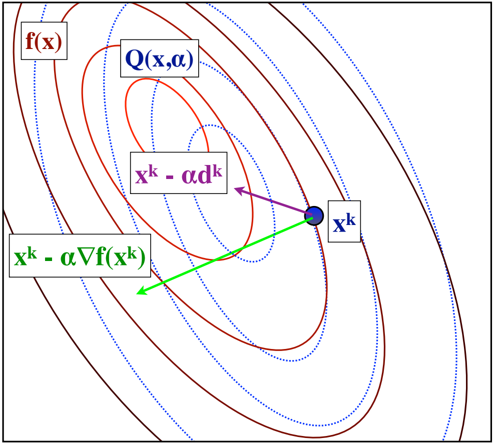

Convergence Rate of Newton's Method

- If $\nabla^2 f(x)$ is Lipschitz-continuous and $\nabla^2 f(x) \succeq \mu$, then close to x^* Newton's method has local superlinear convergence:

$$f(x^{t+1}) - f(x^*) \leq \rho_t [f(x^t) - f(x^*)],$$

with $\lim_{t\to\infty} \rho_t = 0$.

- Converges very fast, use it if you can!
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- But requires solving $\nabla^2 f(x)d = \nabla f(x)$.
- Get global rates under various assumptions (cubic-regularization/accelerated/self-concordant).

Newton's Method: Practical Issues

There are many practical variants of Newton's method:

- Modify the Hessian to be positive-definite.
- Only compute the Hessian every m iterations.
- Only use the diagonals of the Hessian.
- Quasi-Newton: Update a (diagonal plus low-rank) approximation of the Hessian (BFGS, L-BFGS).

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- Hessian-free: Compute d inexactly using Hessian-vector products:
$$\nabla^2 f(x)d = \lim_{\delta \to 0} \frac{\nabla f(x + \delta d) - \nabla f(x)}{\delta}$$
- Barzilai-Borwein: Choose a step-size that acts like the Hessian over the last iteration:
$$\alpha = \frac{(x^+ - x)^T(\nabla f(x^+) - \nabla f(x))}{\|\nabla f(x^+) - f(x)\|^2}$$

Another related method is nonlinear conjugate gradient.

Outline

Motivation: Sparse Regularization

- Consider ℓ_1-regularized optimization problems,

$$\min_x f(x) + \lambda \|x\|_1,$$

  where f is differentiable.
- For example, ℓ_1-regularized least squares,

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- Regularizes and encourages sparsity in x
- The objective is non-differentiable when any $x_i = 0$.
- How can we solve non-smooth convex optimization problems?

Sub-Gradients and Sub-Differentials

Recall that for *differentiable* convex functions we have

$$f(y) \geq f(x) + \nabla f(x)^T (y - x), \forall x, y.$$

A vector d is a subgradient of a convex function f at x if

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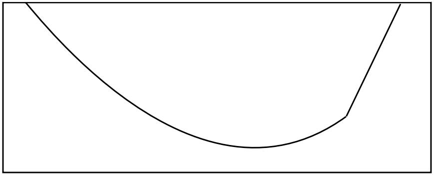

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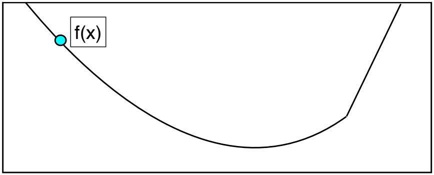

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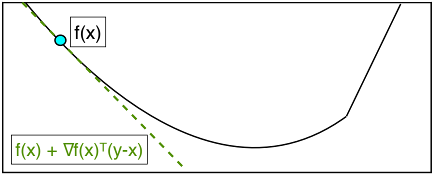

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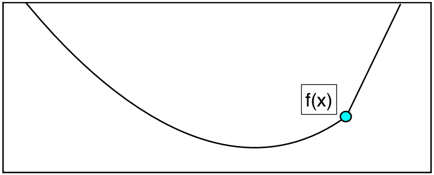

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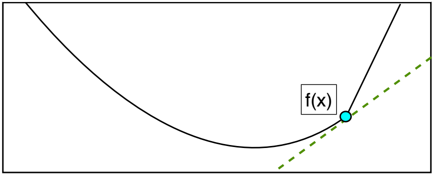

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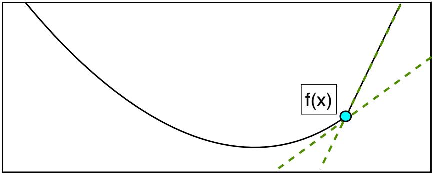

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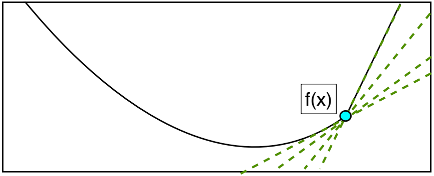

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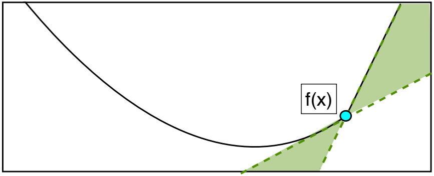

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- f is differentiable at x iff $\nabla f(x)$ is the only subgradient.
- At non-differentiable x, we have a set of subgradients.
- Set of subgradients is the sub-differential $\partial f(x)$.
- Note that $0 \in \partial f(x)$ iff x is a global minimum.

Sub-Differential of Absolute Value and Max Functions

- The sub-differential of the absolute value function:

$$\partial|x| = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ [-1, 1] & x = 0 \end{cases}$$

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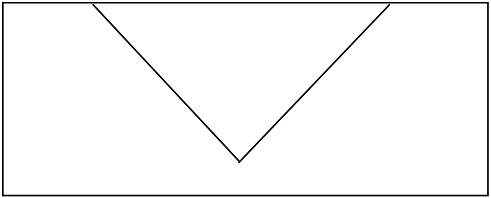

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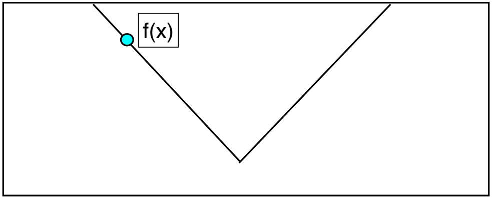

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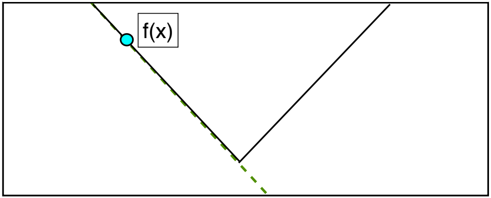

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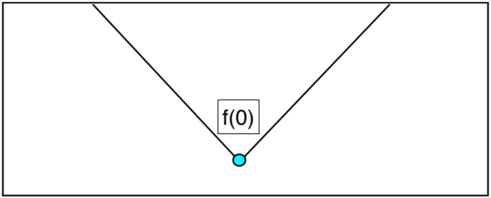

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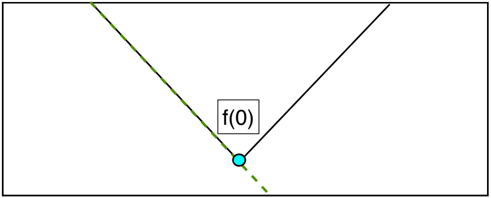

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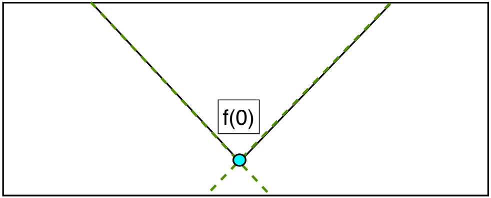

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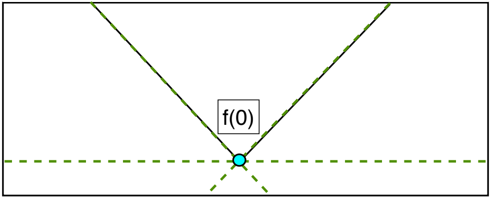

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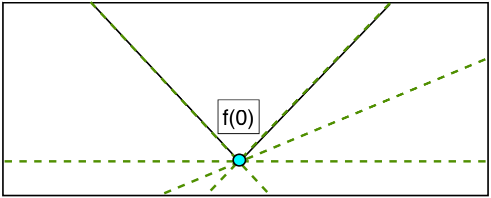

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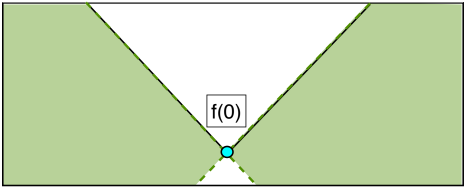

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- The sub-differential of the maximum of differentiable f_i:

$$\partial \max\{f_1(x), f_2(x)\} = \begin{cases} \nabla f_1(x) & f_1(x) > f_2(x) \\ \nabla f_2(x) & f_2(x) > f_1(x) \\ \theta \nabla f_1(x) + (1-\theta)\nabla f_2(x) & f_1(x) = f_2(x) \end{cases}$$

(any convex combination of the gradients of the argmax)

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- Bad news: Rates are optimal for black-box methods.
- But, we often have more than a black-box:
  - We can use structure to get faster rates than black-box methods.

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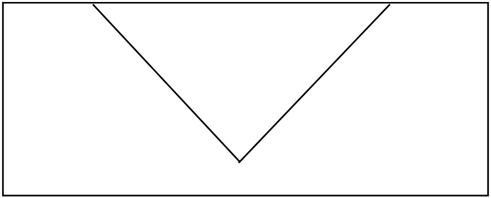

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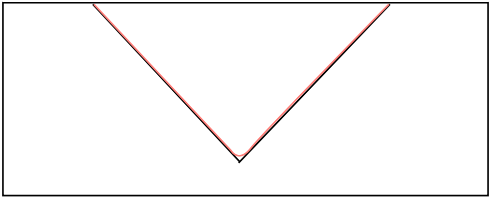

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- Generic smoothing strategy: strongly-convex regularization of convex conjugate.[Nesterov, 2005]

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- You can get the $O(1/t)$ rate for $\min_x \max\{f_i(x)\}$ for f_i convex and smooth using *mirror-prox* method.[Nemirovski, 2004]
    - See also Chambolle & Pock [2010].

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- These are smooth objective with 'simple' constraints.

$$\min_{x \in \mathcal{C}} f(x).$$

Optimization with Simple Constraints

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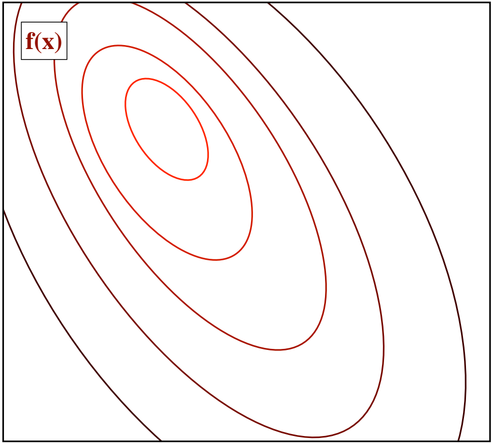

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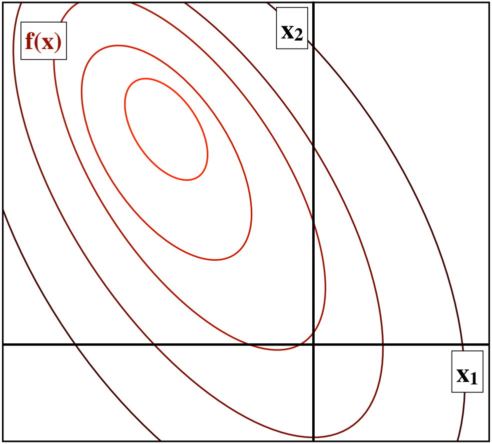

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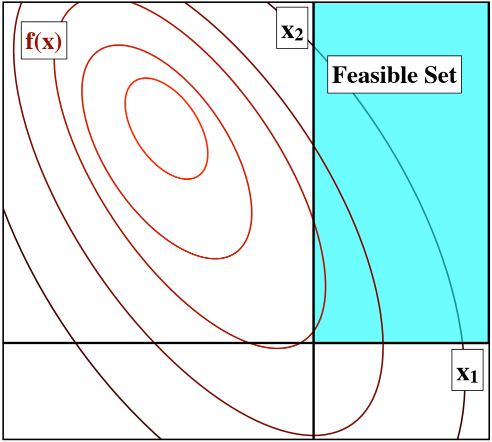

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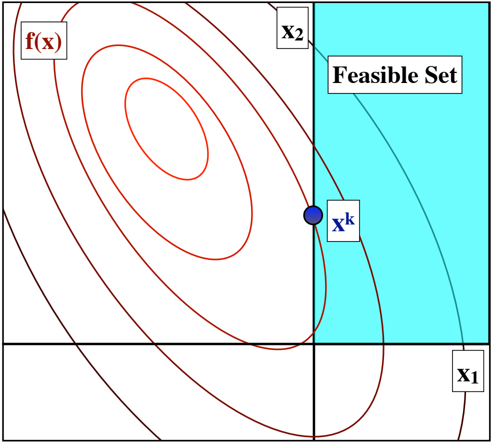

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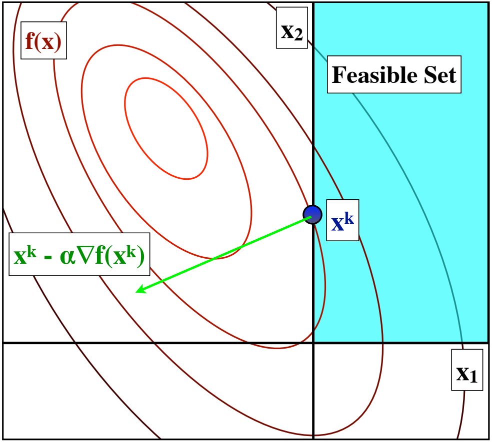

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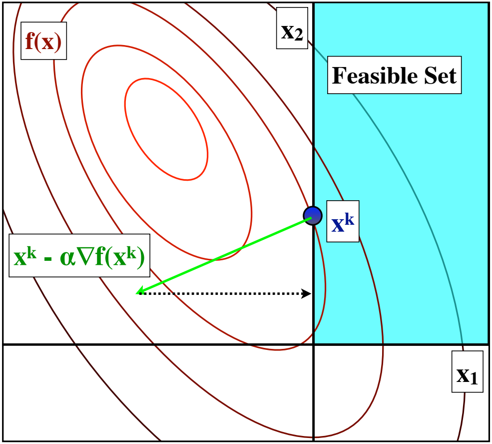

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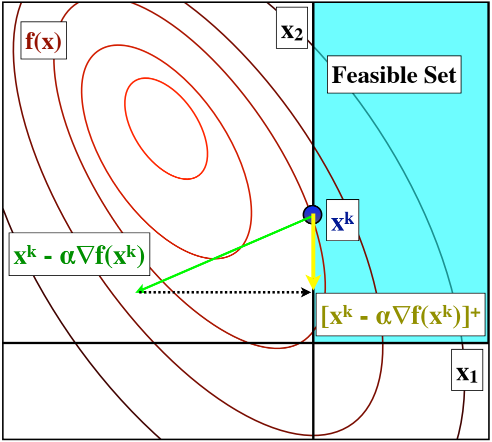

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- Intersection of simple sets: Dykstra's algorithm.

Convex Optimization Zoo

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- For Newton, you need to project under $\|\cdot\|_{\nabla^2 f(x)}$

  (expensive, but special tricks for the case of simplex or lower/upper bounds)
- You don't need to compute the projection exactly.

Proximal-Gradient Method

- A generalization of projected-gradient is Proximal-gradient.
- The proximal-gradient method addresses problem of the form

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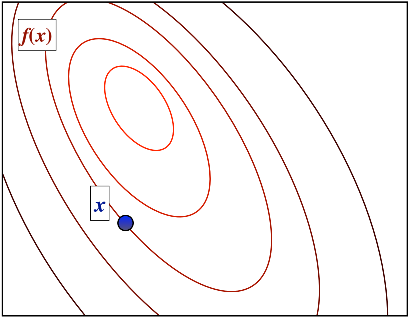

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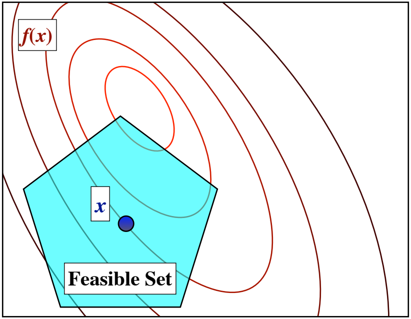

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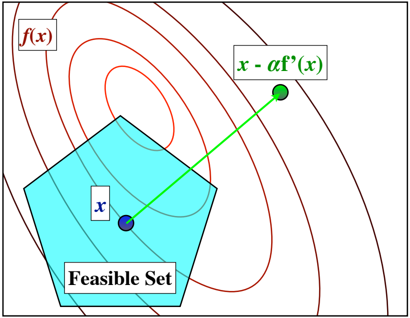

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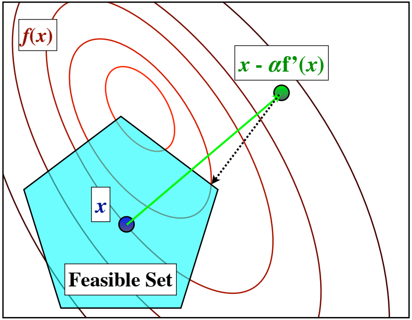

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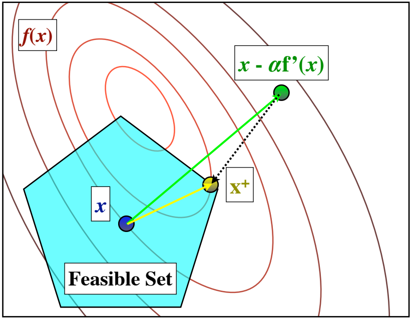

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- But for many problems we can not efficiently compute this operator.

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- Do inexact methods have the same rates?
  - *Yes, if the errors are appropriately controlled*. [Schmidt et al., 2011]

Convergence Rate of Inexact Proximal-Gradient

Proposition *[Schmidt et al., 2011] If the sequences of gradient errors* $\{||e_t||\}$ *and proximal errors* $\{\sqrt{\varepsilon_t}\}$ *are in* $\{O((1-\mu/L)^t)\}$*, then the inexact proximal-gradient method has an error of* $O((1-\mu/L)^t)$*.*

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- Classic result as a special case (constants are good).
- The rates degrades gracefully if the errors are larger.
- Similar analyses in convex case.
- Huge improvement in practice over black-box methods.
- Also exist accelerated and spectral proximal-gradient methods.

Discussion of Proximal-Gradient

- Theoretical justification for what works in practice.
- Significantly extends class of tractable problems.
- Many applications with inexact proximal operators:
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- But, it assumes computing $\nabla f(x)$ and $\text{prox}_h[x]$ have similar cost.
- Often $\nabla f(x)$ is much more expensive:
  - We may have a large dataset.
  - Data-fitting term might be complex.
- Particularly true for structured output prediction:
  - Text, biological sequences, speech, images, matchings, graphs.

Costly Data-Fitting Term, Simple Regularizer

- Consider fitting a conditional random field with ℓ_1-regularization:

$$\min_{x \in \mathbb{R}^P} \quad \underbrace{\frac{1}{N}\sum_{i=1}^{N} f_i(x)}_{\text{costly smooth}} + \underbrace{r(x)}_{\text{simple}}$$

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- Inspiration from the smooth case:
  - For smooth high-dimensional problems, L-BFGS quasi-Newton algorithm outperforms accelerated/spectral gradient methods.

Quasi-Newton Methods

- Gradient method for optimizing a smooth f:

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Quasi-Newton Methods

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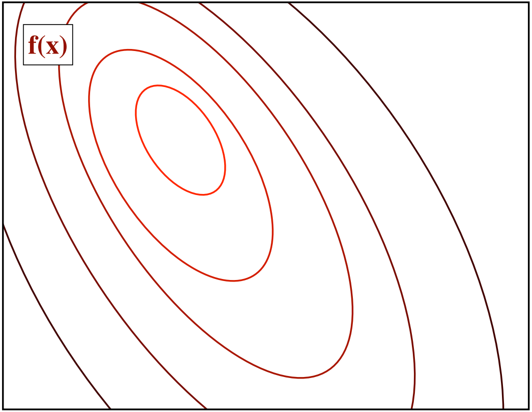
- H approximates the second-derivative matrix.
- L-BFGS is a particular strategy to choose the H values:
  - Based on gradient differences.
  - Linear storage and linear time.

`http://www.di.ens.fr/~mschmidt/Software/minFunc.html`

Gradient Method and Newton's Method



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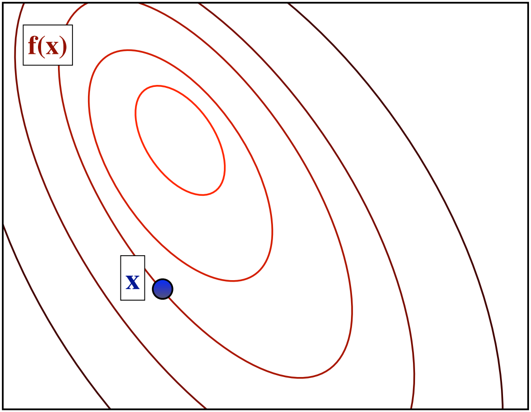

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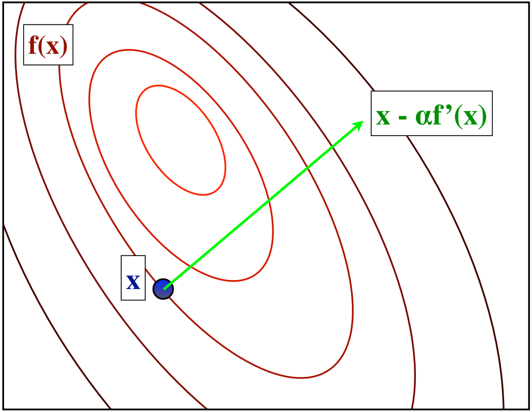

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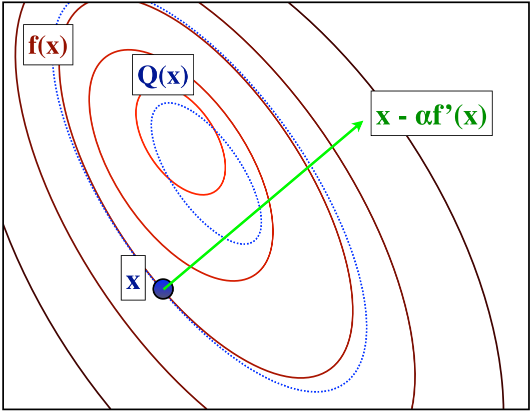

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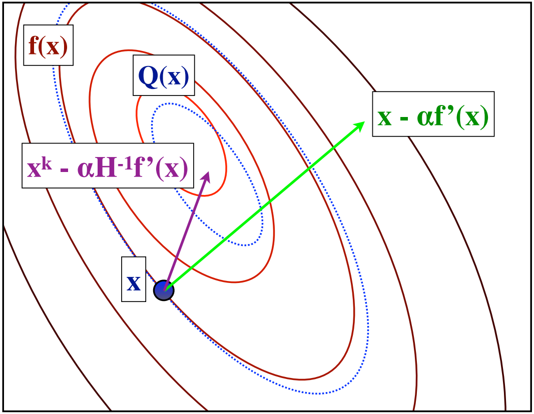

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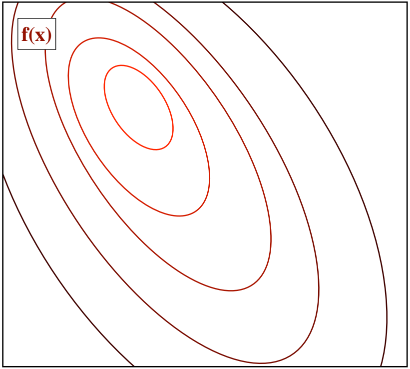

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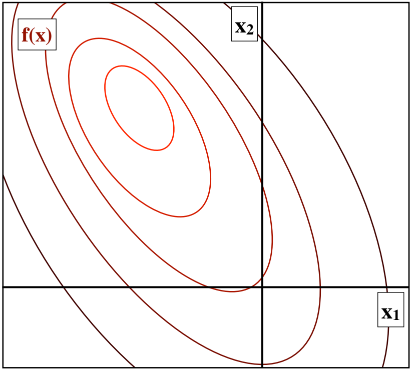

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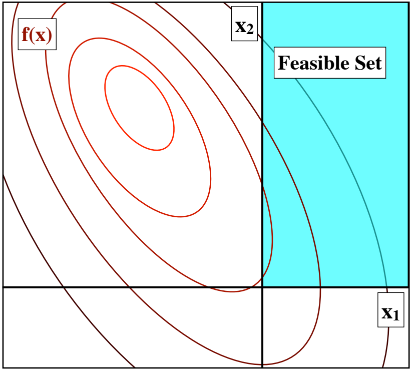

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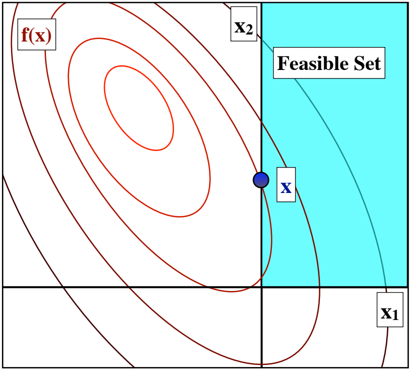

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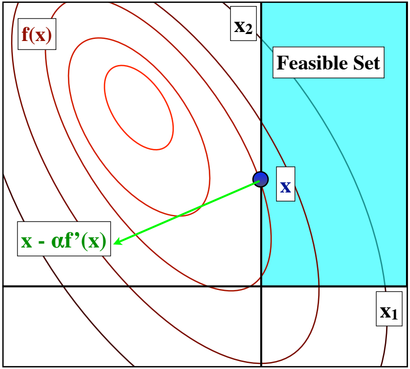

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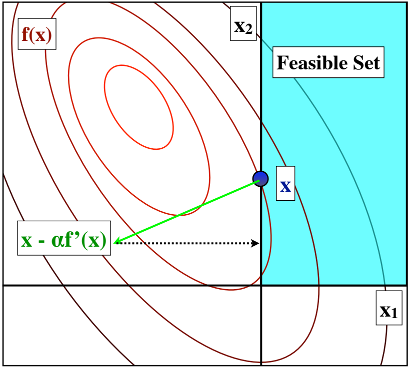

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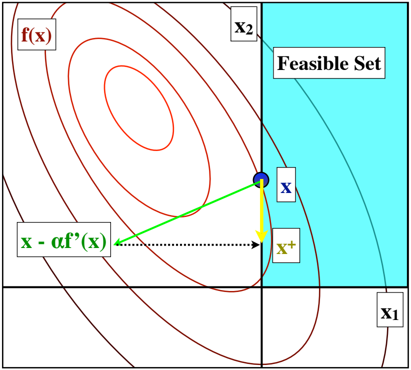

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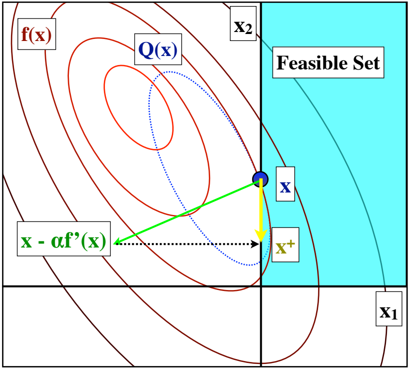

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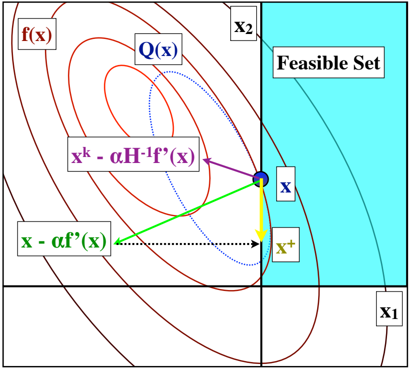

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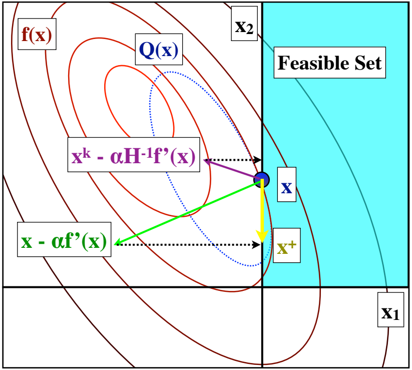

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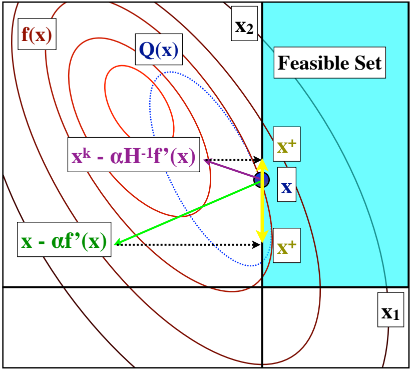


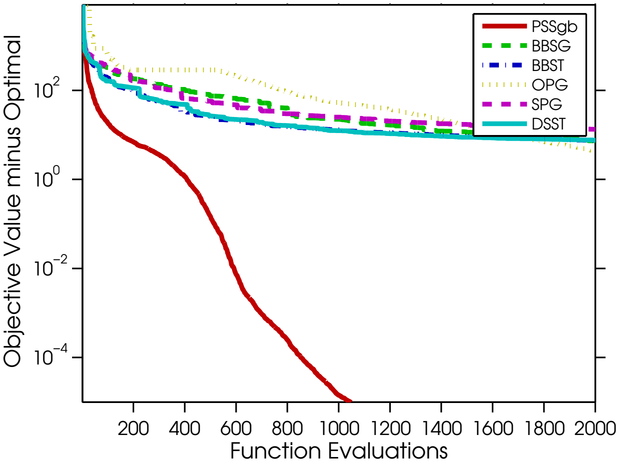
Two-Metric (Sub)Gradient Projection

- In some cases, we can modify H to make this work:
  - Bound constraints.
  - Probability constraints.
  - L1-regularization.
- Two-metric (sub)gradient projection.

  [Gafni & Bertskeas, 1984, Schmidt, 2010].
- Key idea: make H diagonal with respect to coordinates near non-differentiability.

Comparing to accelerated/spectral/diagonal gradient

Comparing to methods that do not use L-BFGS (sido data):



http://www.di.ens.fr/~mschmidt/Software/L1General.html

Inexact Proximal-Newton

- The broken proximal-Newton method:

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$$\mathrm{prox}_r[y] = \underset{x \in \mathbb{R}^P}{\arg\min}\ r(x) + \frac{1}{2}\|x - y\|^2,$$

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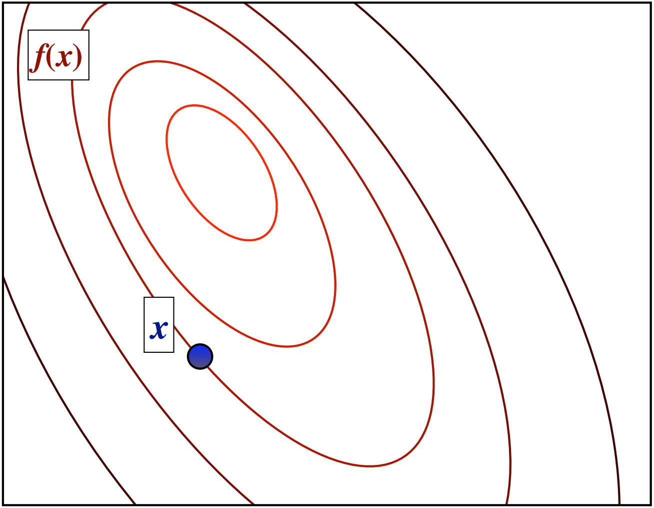
- But, the prox is expensive even with a simple regularizer.
- Solution: use a cheap approximate solution.

(e.g., spectral proximal-gradient)

Inexact Projected Newton



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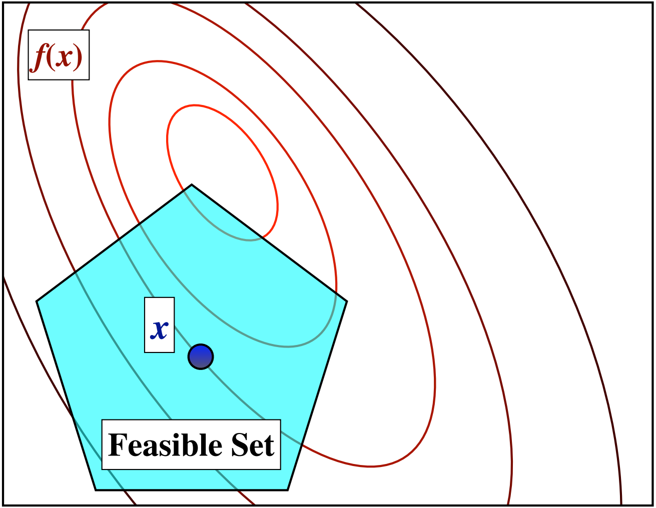

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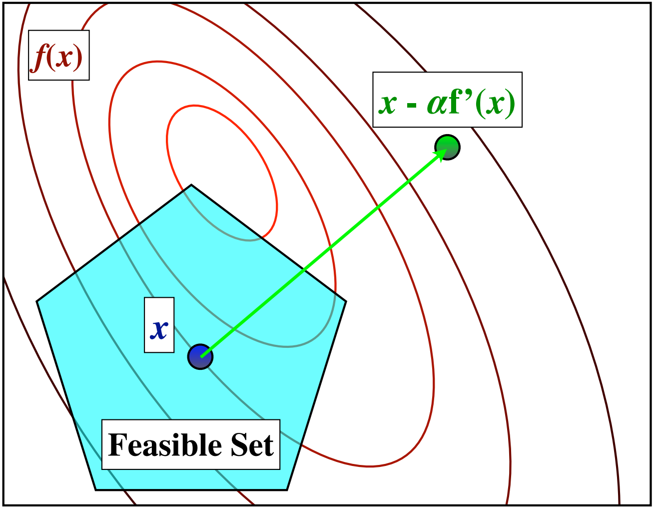

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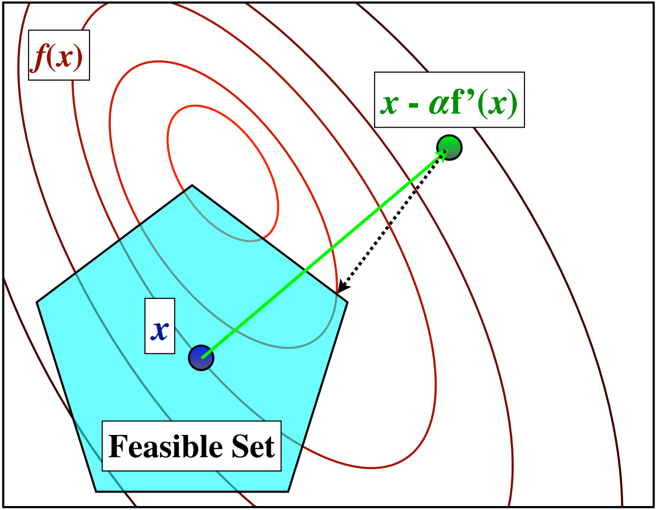

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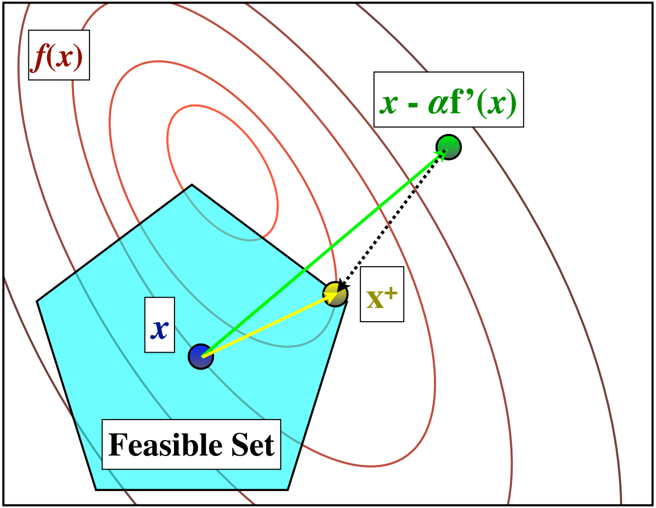

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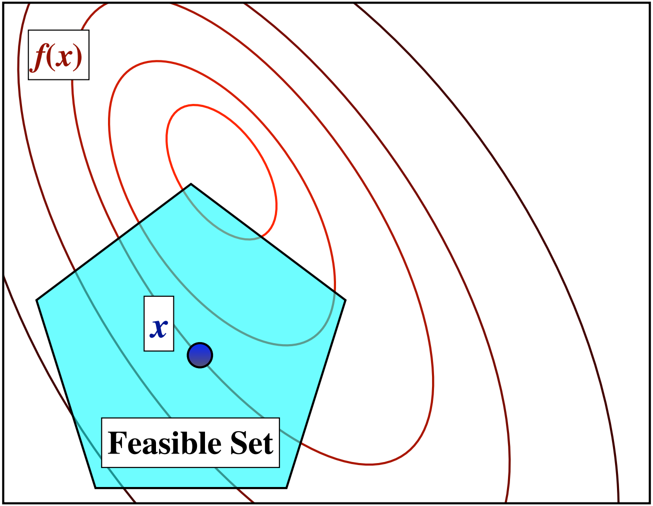

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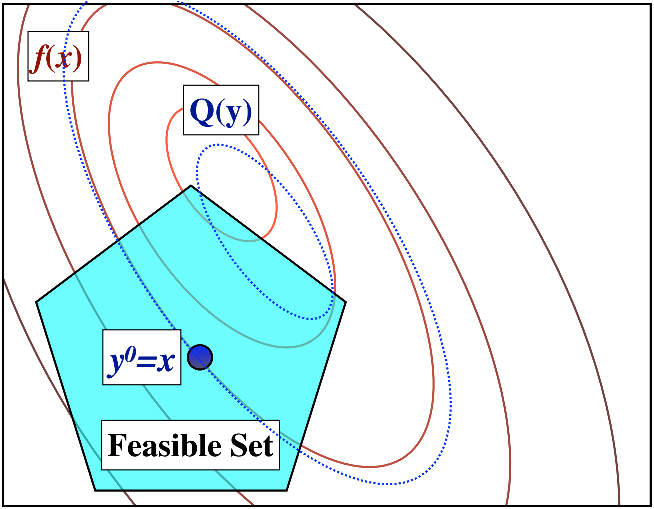

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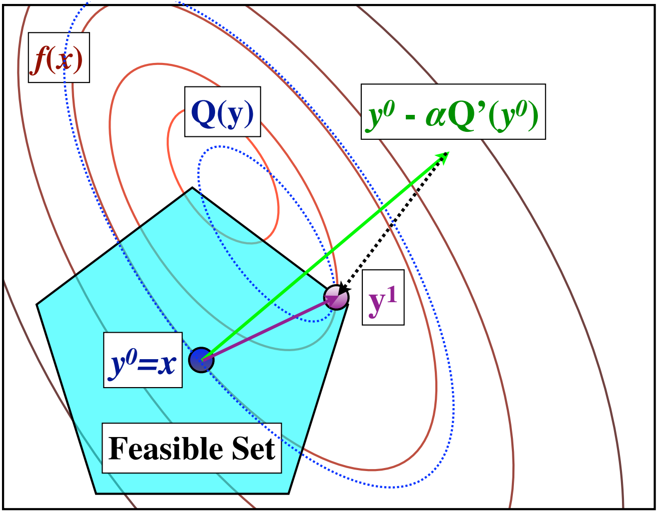

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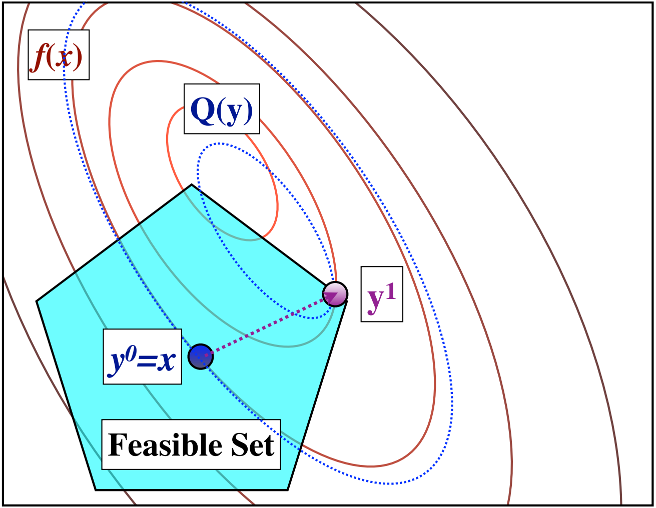

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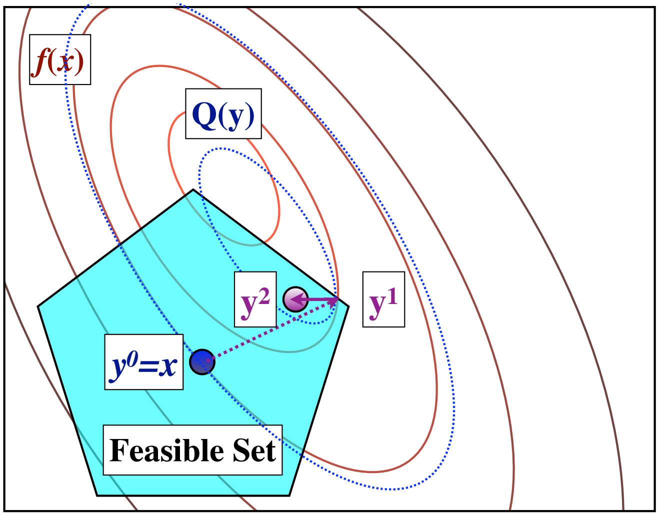

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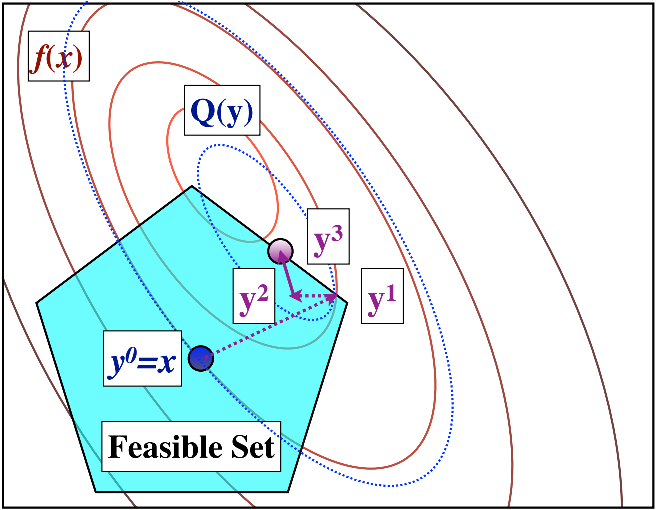

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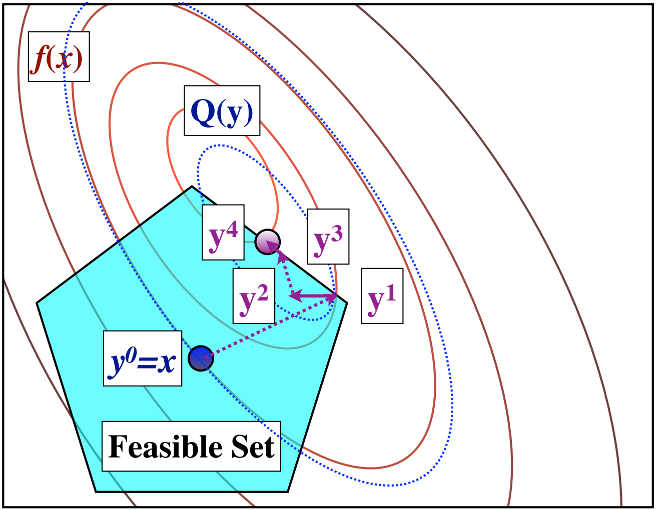

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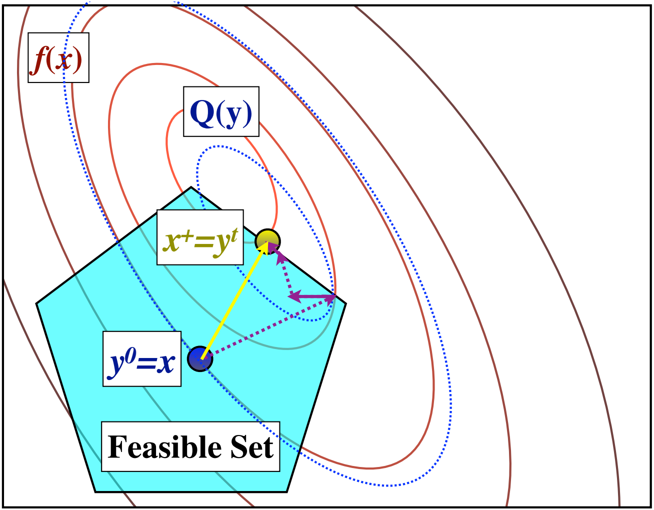

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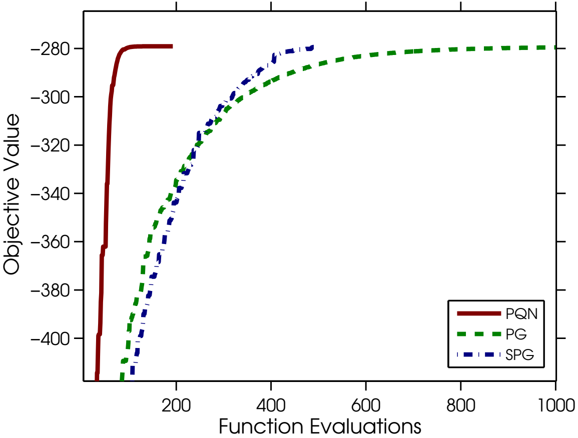
- "Optimizing costly functions with simple regularizers".

Graphical Model Structure Learning with Groups

Comparing PQN to first-order methods on a graphical model structure learning problem. [Gasch et al., 2000, Duchi et al., 2008].



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- If prox can not be computed exactly: Linearized ADMM.

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- SVM non-smooth strongly-convex primal:

$$\min_x C \sum_{i=1}^{N} \max\{0, 1 - b_i a_i^T x\} + \frac{1}{2}\|x\|^2.$$

- SVM smooth dual:

$$\min_{0 \leq \alpha \leq C} \frac{1}{2}\alpha^T A A^T \alpha - \sum_{i=1}^{N} \alpha_i$$

- Smooth bound constrained problem:
  - Two-metric projection (efficient Newton-liked method).
  - Randomized coordinate descent (next section).

Discussion

- State of the art methods consider several other issues:
  - Shrinking: Identify variables likely to stay zero. [El Ghaoui et al., 2010].
  - Continuation: Start with a large λ and slowly decrease it. [Xiao and Zhang, 2012]
  - Frank-Wolfe: Using linear approximations to obtain efficient/sparse updates.

Frank-Wolfe Method

- In some cases the projected gradient step

$$x^+ = \operatorname*{arg\,min}_{y \in \mathcal{C}} \left\{ f(x) + \nabla f(x)^T (y - x) + \frac{1}{2\alpha} \|y - x\|^2 \right\},$$

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- Iterate can be written as convex combination of vertices of $\mathcal{C}$.
- $O(1/t)$ rate for smooth convex objectives, some linear convergence results for smooth and strongly-convex.[Jaggi, 2013]

Alternatives to Quadratic/Linear Surrogates

- Mirror descent uses the iterations[Beck & Teboulle, 2003]

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  where $\mathcal{D}$ is a Bregman-divergence:
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- Mairal [2013,2014] considers general surrogate optimization:

$$x^+ = \arg\min_{y \in \mathcal{C}} \{g(y)\},$$

  where g upper bounds f, $g(x) = f(x)$, $\nabla g(x) = \nabla f(x)$, and $\nabla g - \nabla f$ is Lipschitz-continuous.
  - Get $O(1/k)$ and linear convergence rates depending on $g - f$.

Outline

Big-N Problems

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- Simple example is least-squares,

$$f_i(x) := (a_i^T x - b_i)^2.$$

- Other examples:
  - logistic regression, Huber regression, smooth SVMs, CRFs, etc.

Stochastic vs. Deterministic Gradient Methods

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  - As in subgradient method, we require $\alpha_t \to 0$.
  - Classical choice is $\alpha_t = O(1/t)$.

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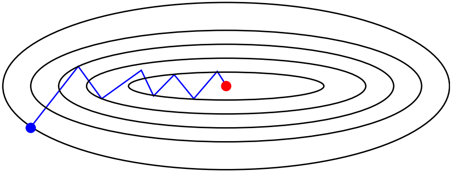

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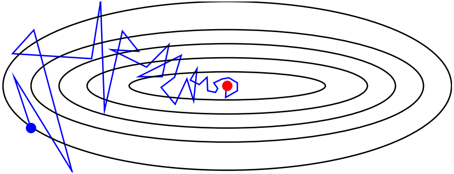

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- Bad news for smooth problems:
  - smoothness does not help stochastic methods.

Algorithm	Assumptions	Exact	Stochastic
Gradient	Convex	$O(1/t)$	$O(1/\sqrt{t})$
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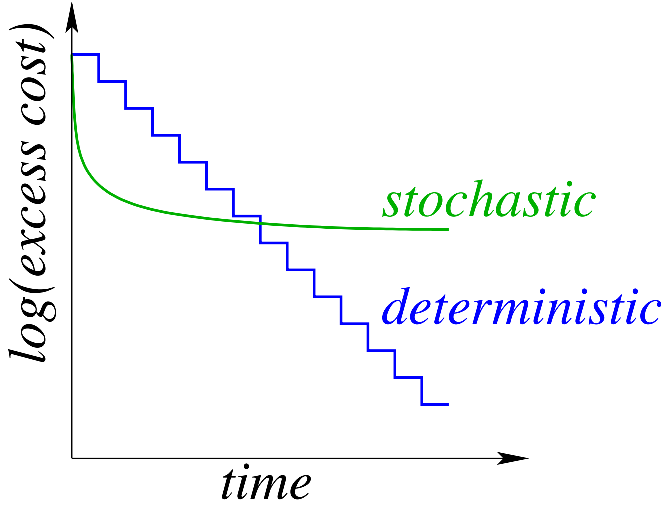
Deterministic vs. Stochastic Convergence Rates

Plot of convergence rates in smooth/strongly-convex case:



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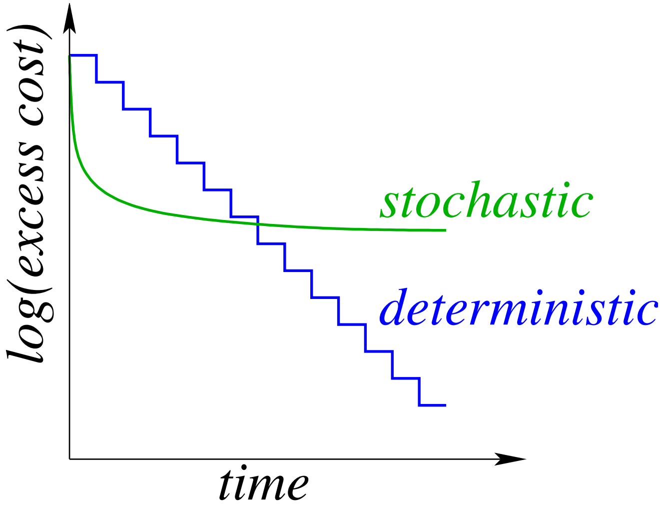
Motivation for Hybrid Methods for Smooth Problems



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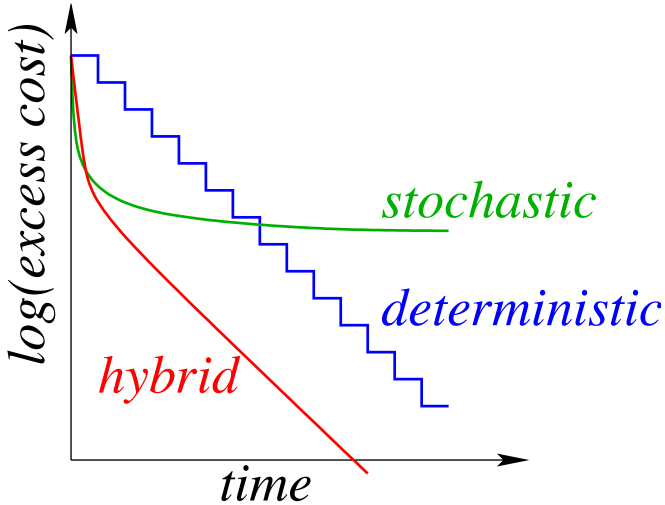

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  - Stochastic variant of increment average gradient (IAG). [Blatt et al., 2007]
  - Assumes gradients of non-selected examples don't change.
  - Assumption becomes accurate as $||x^{t+1} - x^t|| \to 0$.
  - Memory requirements reduced to $O(N)$ for many problems.

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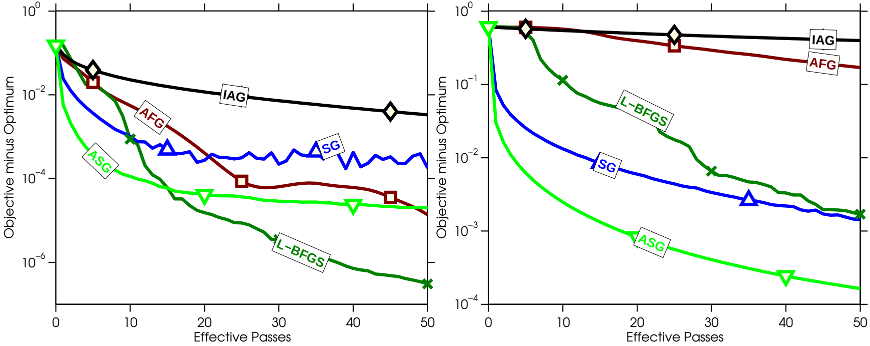
- Recent work extends this result in various ways:
  - Similar rates for stochastic dual coordinate ascent. [Shalev-Schwartz & Zhang, 2013]
  - Memory-free variants. [Johnson & Zhang, 2013, Madavi et al., 2013]
  - Proximal-gradient variants. [Mairal, 2013]
  - ADMM variants. [Wong et al., 2013]
  - Improved constants. [Defazio et al., 2014]
  - Non-uniform sampling. [Schmidt et al., 2013]

Comparing FG and SG Methods

- quantum ($n = 50000$, $p = 78$) and rcv1 ($n = 697641$, $p = 47236$)



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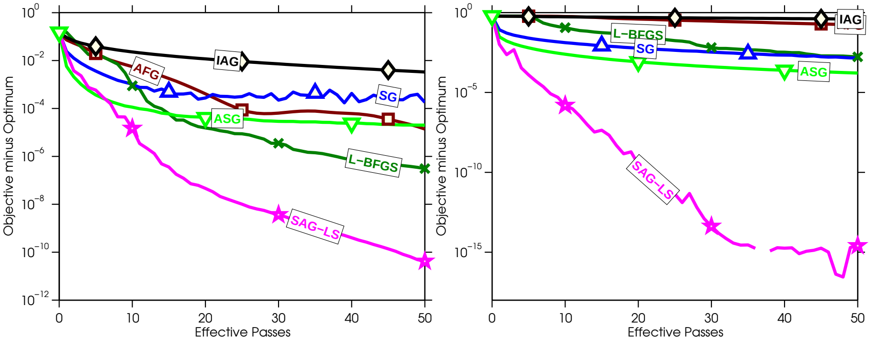

Coordinate Descent Methods

- Consider problems of the form

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  where f and f_{ij} are smooth and $G = \{\mathcal{V}, \mathcal{E}\}$ is a graph.

- Appealing strategy for these problems is coordinate descent:

$$x_j{}^+ = x_j - \alpha \nabla_j f(x).$$

  (i.e., update one variable at a time)

- We can typically perform a cheap and precise line-search for α.

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$$\min_x f(Ax),$$

  where bottleneck is matrix multiplication and A is low-rank.
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- Q formed from Gram-Schmidt and matrix multiplication with random vectors gives very good approximation bounds, if singular values decay quickly.[Halko et al., 2011, Mahoney, 2011]
- May work quite badly if singular values decay slowly.

Outline

1. Convex Functions
2. Smooth Optimization
3. Non-Smooth Optimization
4. Randomized Algorithms
5. Parallel/Distributed Optimization

Motivation for Parallel and Distributed

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- Two recent trends:
  - We aren't making large gains in serial computation speed.
  - Datasets no longer fit on a single machine.
- Result: we must use parallel and distributed computation.
- Two major issues:
  - Synchronization: we can't wait for the slowest machine.
  - Communication: we can't transfer all information.

Embarassing Parallelism in Machine Learning

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- These allow optimal linear speedups.
  - You should always consider this first!

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- You need to decrease step-size in proportion to asynchrony.
- Convergence rate decays elegantly with delay m.[Niu et al., 2011]

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- Only needs to communicate single coordinates.
- Again need to decrease step-size for convergence.
- Speedup is based on density of graph.[Richtarik & Takac, 2013]

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  - Each processor has its own parameter vector x_c.
  - Each processor only communicates with a limited number of neighbours $\text{nei}(c)$.

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$$x_c = \frac{1}{|\text{nei}(c)|} \sum_{c' \in \text{nei}(c)} x_c - \frac{\alpha_c}{M} \sum_{i=1}^{M} \nabla f_i(x_c).$$

- Gradient descent is special case where all neighbours communicate.
- With modified update, rate decays gracefully as graph becomes sparse.[Shi et al., 2014]
- Can also consider communication failures.[Agarwal & Duchi, 2011]

Summary

Summary:

- Part 1: Convex functions have special properties that allow us to efficiently minimize them.
- Part 2: Gradient-based methods allow elegant scaling with problems sizes for smooth problems.
- Part 3: Tricks like proximal-gradient methods allow the same scaling for many non-smooth problems.
- Part 4: Randomized algorithms allow even further scaling for problem structures that commonly arise in machine learning.
- Part 5: The future will require parallel and distributed that are asynchronous and are careful about communication costs.

Thank you for coming and staying until the end!

Bonus Slide: Non-Convex Rates

- For non-convex function we have

$$\|\nabla f(x^j)\|^2 = O(1/t),$$

for at least one j in the sequence.[Ghadimi & Lan, 2013]

- Recall the rate for non-convex optimization with grid-search,

$$f(x) - f(x^*) = O(1/t^{1/d}).$$

Bayesian optimization can improve the dependence on d if the function is sufficiently smooth.[Bull, 2011]